

# Extensions to the Gaussian copula: random recovery and random factor loadings

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*This paper presents two new models of portfolio default loss that extend the standard Gaussian copula model yet preserve tractability and computational efficiency. In one extension, we randomize recovery rates, explicitly allowing for the empirically well-established effect of inverse correlation between recovery rates and default frequencies. In another extension, we build into the model random systematic factor loadings, effectively allowing default correlations to be higher in bear markets than in bull markets. In both extensions, special cases of the models are shown to be as tractable as the Gaussian copula model and to allow efficient calibration to market credit spreads. We demonstrate that the models – even in their simplest versions – can generate highly significant pricing effects such as fat tails and a correlation “skew” in synthetic CDO tranche prices. When properly calibrated, the skew effect of random recovery is quite minor, but the extension with random factor loadings can produce correlation skews similar to the steep skews observed in the market. We briefly discuss two alternative skew models, one based on the Marshall-Olkin copula, the other on a spread-dependent correlation specification for the Gaussian copula.*

## 1 INTRODUCTION

The modeling of portfolio credit risk has been an active area of research in recent years, triggered in part by substantial and well-published credit events, as well as the increasing popularity of portfolio credit derivatives, most notably the synthetic single-tranche CDO. A key feature of portfolio credit models is the mechanism with which default co-dependence between obligors is incorporated. While the existence of default co-dependence is firmly backed up by empirical studies (see, eg, Das *et al*, 2002, and the references therein), it is a subtle subject (see, eg, Lucas, 1995, and Embrechts *et al*, 1999) with a multitude of modeling approaches. In one line of thinking, default of each obligor is modeled as a Poisson process with stochastic intensity, and a mechanism is introduced to let the intensity process of one obligor affect the intensity of another. A typical mechanism involves correlating

Brownian drivers and introducing common Poisson-type shocks to the intensity processes. Papers following this approach include Duffie and Singleton (1999), Duffie and Garleanu (2001), and Jarrow and Yu (2001), to name a few. In the related *exponential model* (see, eg, Duffie and Singleton, 1999; Giesecke, 2003; and the discussion in Section 5) the credit spread of a given obligor is decomposed into idiosyncratic and systematic components, the latter representing the intensities of multi-name simultaneous defaults and responsible for generating default co-dependence.

Another line of models – the structural models – involves a micro-economic representation of the capital structure of the firm, with default being triggered by company assets falling below some threshold representing a fraction of company debt. Examples (for single obligors) can be found in Merton (1974), Leland and Toft (1996), Zhou (1997), and Collin-Dufresne and Goldstein (2001). Portfolio credit risk frameworks based on structural models typically involves correlating the asset process for different firms, as is done for instance in the widely used KMV model (see Zeng and Zhang, 2001).

A third line of co-dependence modeling involves the usage of a so-called *copula function* to “stitch” together known marginal single-obligor default probabilities into a joint default distribution. Copula functions are common in actuarial science and are surveyed in Embrechts *et al* (2001) and Schönbucher (2001). The copula approach is extremely flexible (virtually any model has a copula representation) and is particularly popular with practitioners in CDO markets due to the computational efficiency enjoyed by certain simple classes of copulas. With the focus of this article being firmly on CDO applications, we shall stay inside the copula framework throughout most of this article.

Recall that a synthetic single-tranche CDO is a contract that carves out a certain portion of the credit losses experienced on a pool of standard credit default swaps; see Boscher and Ward (2002) for details. For the valuation of this product, the market standard copula is the *Gaussian copula* (Li, 2000) and its convenient Student *t* extension (Frey and McNeil, 2003). In this model each underlying credit risk is associated with a Gaussian (or Student *t*) random default variable whose value determines the time of default. The map from default variables to default times can be chosen to reproduce observed default swap quotes, allowing an interpretation of the default variables as firm asset values; see Merton (1974). The random default variables are assumed correlated with a constant correlation matrix, the entries of which can conveniently be thought of as correlations of firm asset returns. If the correlation matrix can be represented by a low number of systematic factors, portfolio loss distributions – and thereby CDO tranche prices – can be computed efficiently using either recursions or Fourier methods; see Andersen *et al* (2003) or Gregory and Laurent (2003).

While convenient and intuitive, the Gaussian copula has a number of obvious shortcomings as a model of the real world. For instance, common formulations of the model (Li, 2000, or Schönbucher, 2001) make the assumption that recovery rates on default are known firm-specific constants. In reality, however, forecasting

recovery rates of companies in default is notoriously difficult. While a number of modeling frameworks exist (eg, Gupton and Stein, 2002), empirical recovery rate distributions have high variance and the certainty with which one can predict recovery at present is quite low. Among the few undisputed facts about recovery rates, however, is the observation that average recovery rates tend to be inversely related to default rates: in a bad year, not only are there many defaults but recoveries are also low. The empirical literature supporting this observation is large and growing; relevant papers include Acharya *et al* (2003), Altman *et al* (2003), Frye (2000), Frye (2003), and Hu and Perraudin (2002), to name a few. Altman *et al* (2003) reviews much of the empirical research in detail. We should note that the precise meaning of the term “recovery rate” depends on the context. For credit derivatives, recovery rates are contractually derived from the market price of outstanding debt shortly after default, whereas in credit risk analysis it may be more suitable to use the liquidation recovery rates determined in the workout process.

Models for stochastic recovery rates have been proposed by a number of authors – see Frye (2000), Jarrow (2001), Jokivuolle and Peura (2000), and Pykhtin (2003). The primary focus of most of the existing literature is on applications to risk management and the management of tail risk. In this paper, however, we shall tailor the modeling of recovery rates to the pricing of synthetic CDOs and other structured credit derivatives. This, in turn, imposes a number of constraints. First, precision in the fit-to-market observables now becomes of prime importance, necessitating the construction of explicit calibration routines. Second, to efficiently risk-manage and hedge synthetic CDO positions, tractability and numerical efficiency is critical. Third, given the nature of the underlying collateral (credit default swaps), recovery rates as a percentage of notional are bounded on  $[0, 1]$  and should be modeled as such.<sup>1</sup> This paper introduces, and embeds into a Gaussian copula, a multi-dimensional factor-type recovery model that satisfies these constraints.

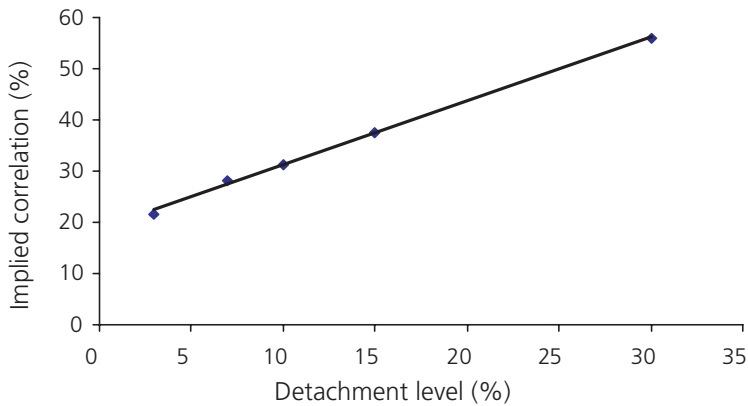
With the introduction of a quoted market on standardized tranches of credit indices, another imperfection of the Gaussian copula has recently become evident: the allocation of value across tranches implied by the model does not match the observable market. This effect is perhaps best discussed in the context of implied “base correlation”, defined as the constant correlation (for all pairs of credits) required for a Gaussian copula to match the price of an all-equity CDO tranche with fixed upper detachment level.<sup>2</sup> Graphed against the upper detachment level, the implied base correlations observable in the market tend to form a steeply increasing skew, as illustrated in Figure 1 for five-year tranches on the I-Boxx index.

Equipped with interpolation and/or extrapolation rules, one can use graphs

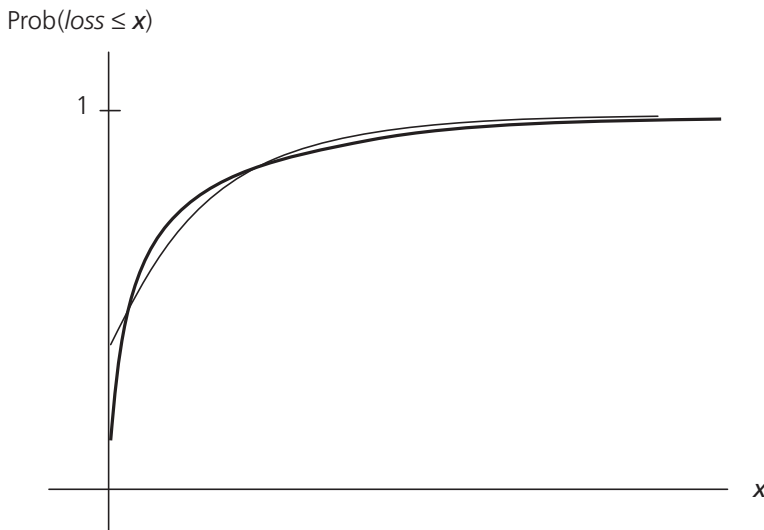
<sup>1</sup> In Frye (2000) and Pykhtin (2003), for example, loss amounts are Gaussian and log-normal, respectively, and thus unbounded.

<sup>2</sup> That is, we consider loss tranches spanning the first  $x\%$  of the total pool notional. Non-equity tranches spanning a percentage loss interval of, say,  $[l, u]$ ,  $0 < l < u$ , can trivially be priced as differences of two equity tranches with  $x = u$  and  $x = l$ , respectively.

**FIGURE 1** Base correlation skew in I-Boxx NA (May 2004).



**FIGURE 2** Market implied portfolio loss distribution (schematic).



Schematic portfolio loss distribution in the presence of an upward-sloping base correlation skew (fat curve). The loss distribution for a Gaussian copula is given for reference (thin curve).

such as that in Figure 1 to price arbitrary tranches on standard indices. Apart from the fact that such schemes may be subject to arbitrage,<sup>3</sup> this practice has obvious shortcomings. For instance, for customized portfolios with composition and/or maturities different from that of a traded index it is unclear how to use the base correlation information in a systematic way. It is also unclear how to use the market information for the pricing of non-vanilla structured instruments, such as

<sup>3</sup> Arbitrages will occur either if the base correlation skew is too steep or if it has gradients that change too rapidly (as will happen with linear interpolation, say).

CDOs on portfolios of CDOs (“CDO-squared”). To address these issues and to predict the evolution of the skew with changes in spreads as required for hedge computations, we need a fundamental model mechanism capable of producing the correlation skew.

To understand the demands on a model for the base correlation skew, consider that tranche prices are essentially integrals over the risk-neutral portfolio loss distribution, allowing us to back out the market-implied loss distributions from base correlation skew graphs such as that of Figure 1. As demonstrated schematically in Figure 2, the market loss distribution is more “kinked” than that of a Gaussian copula with empirical correlations, with a fat upper tail and a relatively low probability of generating small losses. The fat upper tail represents an increased probability of high losses (that is, a large number of defaults) and makes the senior tranches relatively risky and consequently transact at a high implied base correlation. At the same time, the low probability of generating small losses also represents increased risk for a junior tranche (which absorbs the first few losses); the junior tranche will thereby transact at a relatively low implied base correlation.<sup>4</sup>

While it is easy to make a Gaussian copula fat-tailed by switching to a Student  $t$  copula (or by raising all correlations), what is required is to do so without increasing too much the probability of generating zero losses.<sup>5</sup> Families of copulas exist that can produce market-similar loss distributions directly (see Section 5), but these tend to be unwieldy numerically and difficult to parameterize and make operational. In this paper, we suggest a straightforward extension of the Gaussian copula where factor loadings are made functions of the systematic factors themselves. Interpreting systematic factors as the “state of the market”, we can use this mechanism to mimic the well-known empirical effect that equity (and thereby asset) correlations are higher in a bear market than in a bull market. As we shall show, this mechanism will also induce a strong correlation skew.

The rest of the paper is organized as follows. First, in Section 2, we present a general framework which can accommodate a wide range of factor models. For reference, we introduce the standard Gaussian copula model and briefly review the efficient numerical technique of Andersen *et al* (2003) available for this model. In Sections 3 and 4, we present our models for random recovery and random factor loadings, respectively. For both models, our treatment puts particular emphasis on financial engineering aspects such as model implementation and calibration. For intuition and for traditional portfolio risk management, we also give easy-to-use results for large-portfolio limits. In Section 5 we briefly consider some alternatives to the main developments: first, a model based on the Marshall-Olkin copula,

<sup>4</sup> For a more detailed introduction to CDO tranches and their correlation sensitivity, see, eg, Boscher and Ward (2002).

<sup>5</sup> In the recent paper Hull and White (2004), a partial de-coupling of the upper and lower tails is obtained by using a “double Student  $t$ ” model where residuals and factors are independent Student  $t$  variables. Our tests reveal that this model, while interesting, is too constrained to allow a precise fit to market prices.

and second, the imposition of specific structures on the correlation matrix for the Gaussian copula. Section 6 contains numerical examples and model comparison and we present our conclusions in Section 7. The Appendix collects a number of useful results for Gaussian integrals

## 2 GENERAL FRAMEWORK

Consider a portfolio of  $N$  default-risky obligors. For a fixed time horizon  $[0, T]$ , we associate the obligors with known default probabilities (eg, inferred from default swap quotes)  $p_i(T)$ ,  $i=1, \dots, N$ , for default before time  $T$ . Note that we usually suppress the dependence of default probabilities on the time horizon  $T$ .

On default, the obligors generate (possibly random) loss amounts of  $l_i$ ,  $i=1, \dots, N$ . Define the random default time of obligor  $i$  to be  $\tau_i$ . The total portfolio loss experienced on  $[0, T]$  is then

$$L = \sum_{i=1}^N l_i 1_{\tau_i \leq T} \tag{1}$$

with expected value

$$E(L) = \sum_{i=1}^N p_i E(l_i | \tau_i < T)$$

In our notation the *indicator function*,  $1_{\tau_i \leq T}$ , in (1) is one if the  $i$ th obligor defaults no later than time  $T$  and zero otherwise. We assume that each  $l_i$  is bounded, ie,  $l_i \in [0, l_i^{\max}]$ ,  $l_i^{\max} \in \mathbb{R}_+$ , such that  $0 \leq L \leq \sum l_i^{\max} = L^{\max}$ . Often, it is convenient to introduce the notion of a *recovery rate*,  $R_i$ , whereby

$$l_i = l_i^{\max} (1 - R_i) \tag{2}$$

for (possibly random)  $R_1, \dots, R_N$  taking values on  $[0, 1]$ . To provide further structure to the loss distribution, we assume the existence of continuous random variables  $X_1, \dots, X_N$  and fixed thresholds  $c_1, \dots, c_N$ , such that

$$1_{\tau_i \leq T} \equiv 1_{X_i \leq c_i}, \quad i = 1, \dots, N$$

Let the distribution function of  $X_i$  be  $F_i^X$ . Assuming that  $F_i^X$  is invertible, we have

$$c_i = (F_i^X)^{-1}(p_i), \quad i = 1, \dots, N$$

Often, it is convenient to think of the  $X_i$  and  $c_i$  as proxies for firm asset and liquidation values, respectively, as in the structural model of Merton (1974).

Throughout, we make the assumption that there exists a  $d$ -dimensional vector of independent conditioning variables  $Z = (Z_1, \dots, Z_d)$  such that all components of the augmented vector  $(X_1, \dots, X_N, l_1, \dots, l_N)$  are independent when conditioned on  $Z$ . Without loss of generality we assume that the  $Z_i$ ,  $i = 1, \dots, d$ , all have zero mean and unit variance.

We shall refer to the vector  $Z$  as the *systematic factors* of our model, and write  $p_i(Z) = E(1_{X_i \leq c_i} | Z) = \text{Prob}(X_i \leq c_i | Z)$  and  $l_i(Z) = E(l_i | Z)$ . By conditional independence,

$$E(L) = E \left( \sum_{i=1}^N p_i(Z) l_i(Z) \right)$$

While  $Z$  may simply be an abstract set of latent variables, it is often useful to think of  $Z$  as representing the industry or economy data that characterize the systematic credit risk of the portfolio. Existence of a low-dimensional conditioning variable is often critical to the numerical efficiency of a model. In particular, the portfolio loss distribution can be efficiently computed with the techniques developed in Andersen *et al* (2003) when the dimension of the conditioning variable is low. As shown in, eg, Andersen *et al* (2003), knowledge of the portfolio loss distribution for a grid of horizon times suffices for the purpose of pricing CDO tranches. For valuation purposes it is crucial that the market quotes of default swaps be reproduced. The valuation of the default swap coupon leg is straightforward and standard given the default probabilities for all times to default swap maturity, but the valuation of the default leg needs to be revisited for the general model framework considered here.

Recall that the default leg of a credit default swap pays  $(1 - R_i)$  at the time of default if and only if default happens before swap maturity. If  $P(0, t)$  denotes the risk-free discount factor for time  $t$  we have<sup>6</sup>

$$\begin{aligned} PV_{i, \text{default}}(0) &= E \left[ \int_0^T (1 - R_i) P(0, t) \mathbf{1}_{\tau_i \in [t, t+dt]} \right] \\ &= \int_0^T P(0, t) E[(1 - R_i) | \tau_i = t] p'_i(t) dt \end{aligned} \quad (3)$$

where the prime denotes differentiation. In the second equation we used the definition of conditional expectation to factorize the expectation.

From (3) we see that the valuation of a default swap requires (only) the expectation of recovery *conditioned on default*. Once this is given the default swap value can be computed as usual from default probabilities. As a consequence, the default probabilities may be inferred by standard methods such as, eg, “bootstrapping”, from default swap quotes. Of course, whilst the use of implied (risk-neutral) default probabilities is required for pricing and hedging, the analysis of portfolio credit risk involves the default probabilities under the real-world measure. The latter probabilities may be estimated from historical default data and are typically smaller than their risk-neutral counterparts.

For the special case where recovery rates are not random, ie, each  $R_i$  is just a number in  $[0, 1]$ , the requisite recovery expectations require no calculation, but in

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<sup>6</sup> Here we make the standard assumption of independence between risk-free discounting rates and default.

models with random recovery it is natural to let the conditional recovery expectation be given exogenously. This not only makes calibration simple, it also avoids the possibly nonsensical and certainly non-intuitive concept of “recovery without default”.

REMARK As just demonstrated, the valuation of default swaps under random recovery requires the knowledge of the expectation of recovery conditioned on default taking place at a given time. Sometimes we need or are given the expectation conditioned instead on default taking place *before* a certain time. If

$$\begin{aligned} f_i(t) &:= \mathbb{E} \left[ (1 - R_i) \mid \tau_i \in [t, t + dt] \right] \\ F_i(t) &:= \mathbb{E} \left[ (1 - R_i) \mid \tau_i \leq t \right] \end{aligned}$$

then we note the following general relationships

$$F_i(T) p_i(T) = \int_0^T f_i(t) p_i'(t) dt \quad (4a)$$

$$f_i(t) = F_i(t) + F_i'(t) \frac{p_i(t)}{p_i'(t)} \quad (4b)$$

We note that the special case of horizon-independent expectations

$$f_i(t) \equiv F_i(t) \equiv \text{const}_i, \text{ for all } t$$

is a solution to (4).

## 2.1 Gaussian copula model

In the general framework set out above the Gaussian copula model is described by letting the factors  $Z$  be Gaussian and by assuming the following relationships

$$\left. \begin{aligned} X_i &= a_i \cdot Z + \sqrt{1 - \|a_i\|^2} \varepsilon_i \\ l_i &= l_i^{\max} (1 - R_i) \end{aligned} \right\} i = 1, \dots, N \quad (5)$$

where  $R_i \in [0, 1]$ ,  $i = 1, \dots, N$ , are non-random and  $\varepsilon_i$ ,  $i = 1, \dots, N$ , define an iid sequence of zero-mean, unit-variance Gaussian variables independent of  $Z$ . The *factor loadings*  $a_i$  are  $d$ -dimensional vectors of length less than one. We note that Andersen *et al* (2003) give an efficient algorithm for computing the factor loadings by optimally approximating an arbitrary correlation structure with a low-dimensional factor structure. Clearly, in (5) the conditional independence of the  $X_i$ s and therefore of defaults is manifest.

In Andersen *et al* (2003) quasi-analytical techniques were developed for studying this model starting from the observation that since defaults are independent when conditioned on factors, we can build the conditional distribution of losses

over the time interval  $[0, T]$  by a simple recursion. To do this we require an arbitrary *loss unit*,  $u$ , to be defined such that the loss amounts  $l_i$  (and thereby all portfolio losses) can be well-approximated<sup>7</sup> by integer multiples of  $u$ , say  $l_i = k_i u$ . Now let  $L_n$ ,  $1 \leq n < N$ , be the loss (measured in loss units) over  $[0, T]$  in the subportfolio consisting of the first  $n$  obligors in some arbitrary ordering. Evidently we have the following recursive relation between the conditional distributions of  $L_n$  and  $L_{n+1}$ :

$$\begin{aligned} \text{Prob}(L_{n+1} = K | Z) &= p_{n+1}(Z) \text{Prob}(L_n = K - k_{n+1} | Z) \\ &\quad + (1 - p_{n+1}(Z)) \text{Prob}(L_n = K | Z) \end{aligned} \quad (6)$$

and we may use this to compute the loss distribution of the portfolio from the boundary case of the empty portfolio for which we have  $\text{Prob}(L_0 = K | Z) = \delta_{K,0}$ . For sensitivity computations it is useful to remark that clearly the portfolio loss distribution cannot depend on the ordering of the obligors.

In general the distribution of a sum of independent random variables is given by the convolution of the distributions of each random variable; the recursion method can be seen as a particular way of performing the (discrete) convolution of the single-obligor loss distributions to obtain the portfolio loss distribution. An alternative is the use of Fourier transformation techniques (see Gregory and Laurent, 2003), although this was found in Andersen *et al* (2003) to be less computationally efficient for the Gaussian copula model.

As shown in Andersen *et al* (2003), sensitivities of expectations over the portfolio loss distribution can be efficiently computed using (6). Let  $X(L)$  be some function of the portfolio loss and consider its sensitivities to default probabilities,<sup>8</sup> ie,  $\partial E(X)/\partial p_i$ . These can be computed as

$$\begin{aligned} \frac{\partial E(X)}{\partial p_i} &= \int \frac{\partial E(X|Z)}{\partial p_i} d\Phi_d(Z) \\ &= \int \frac{dp_i(Z)}{dp_i} \frac{\partial E(X|Z)}{\partial p_i(Z)} d\Phi_d(Z) \\ &= \int \frac{dp_i(Z)}{dc_i} \left( \frac{dp_i}{dc_i} \right)^{-1} \frac{\partial E(X|Z)}{\partial p_i(Z)} d\Phi_d(Z) \end{aligned} \quad (7)$$

where  $\Phi_d$  is the  $d$ -dimensional cumulative Gaussian distribution function with all correlations zero. Here the first two factors of the integrand are easy to compute analytically and the last factor follows from (6):

<sup>7</sup> In principle, any accuracy can be obtained by choosing a sufficiently small loss unit, but accuracy comes at a price since the computational burden is roughly inversely proportional to the loss unit. In practice, for a typical portfolio, rounding errors can be kept small without sacrificing numerical efficiency. See Andersen *et al* (2003) for discussion and a specific algorithm for setting the loss unit.

<sup>8</sup> These are related to *hazard rate* sensitivities and *credit spread deltas* by simple Jacobian factors.

$$\begin{aligned} \frac{\partial E(X|Z)}{\partial p_i(Z)} &= \sum_K X(K) \frac{\partial \text{Prob}(L_N = K | Z)}{\partial p_i(Z)} \\ &= \sum_K X(K) \left[ \text{Prob}(L_{N-1}^{(i)} = K - k_i | Z) - \text{Prob}(L_{N-1}^{(i)} = K | Z) \right] \end{aligned}$$

Here  $L_{N-1}^{(i)}$  is the loss of the portfolio with the  $i$ th obligor removed and we may obtain the distribution of  $L_{N-1}^{(i)}$  by solving (6) recursively for given portfolio loss distribution.

### 3 RANDOM RECOVERY

In this section we consider a special case of the general framework of the previous section in which the loss amounts – or equivalently, the recovery rates – associated with defaults are random. To specify the model let  $a_i, b_i, i = 1, \dots, N$ , be  $d$ -dimensional vectors with  $a_i$  non-negative and of less than unit length. Also, let  $\varepsilon_i, i = 1, \dots, N$ , define a sequence of independent, zero-mean, unit variance random variables independent of  $Z$ . Finally, let  $\xi_i, i = 1, \dots, N$ , be a sequence of independent zero-mean random variables with variances  $\sigma_{\xi_i}^2$ , independent of  $Z$  and the  $\varepsilon_i$ s.

Now consider the model

$$\left. \begin{aligned} X_i &= a_i \cdot Z + \sqrt{1 - \|a_i\|^2} \varepsilon_i \\ l_i &= l_i^{\max} \left( 1 - C_i(\mu_i + b_i \cdot Z + \xi_i) \right) \end{aligned} \right\} i = 1, \dots, N \tag{8}$$

where the  $C_i : \mathbb{R} \rightarrow [0,1], i = 1, \dots, N$ , are arbitrary mapping functions, and the  $\mu_i, i = 1, \dots, N$ , are constants. In this model the  $i$ th recovery is split into (a transformation of) a systematic factor-term ( $b_i \cdot Z$ ) as well as an idiosyncratic one ( $\xi_i$ ). Notice that we thereby retain a factor-structure with recovery rates being influenced by the same economy-wide factors ( $Z$ ) as the default variables, modeling the empirical phenomenon that both default and recovery are affected by the state of the economy (see the discussion in Section 1). The shared set of systematic factors allows us to control through the factor loadings  $a_i$  and  $b_i$  the co-dependence between default and recovery. Specifically, the co-dependence between default of the  $i$ th obligor and recovery on the  $j$ th obligor is controlled by  $a_i \cdot b_j$ , with  $a_i \cdot b_j = 0$  implying independence.

We note that the conditional default probabilities are straightforward:

PROPOSITION 1 If  $F_i^\varepsilon$  is the distribution functions of  $\varepsilon_i$ , then

$$p_i(Z) = F_i^\varepsilon \left( \frac{\left( F_i^X \right)^{-1} (p_i) - a_i \cdot Z}{\sqrt{1 - \|a_i\|^2}} \right) \tag{9}$$

PROOF Simply note that

$$\begin{aligned}
 p_i(Z) &= \text{Prob}(X_i \leq c_i | Z) = \text{Prob}\left(\varepsilon_i \leq \frac{c_i - a_i \cdot Z}{\sqrt{1 - \|a_i\|^2}} \mid Z\right) \\
 &= F_i^\varepsilon\left(\frac{c_i - a_i \cdot Z}{\sqrt{1 - \|a_i\|^2}}\right) = F_i^\varepsilon\left(\frac{(F_i^X)^{-1}(p_i) - a_i \cdot Z}{\sqrt{1 - \|a_i\|^2}}\right)
 \end{aligned}
 \quad \square$$

As discussed in Section 1, there is strong empirical support for dependence of default frequencies and recovery rates. As we shall see later, this phenomenon is of key importance to the tail-behavior of portfolio loss distributions and is worth quantifying. Let us then consider characterizing the correlation structure of the setup in (8). Given the non-linear dependence of recovery rates on the stochastic drivers, we wish to focus on rank correlation, rather than the usual linear (Pearson) correlation coefficient; see, eg, Embrechts *et al* (1999). Specifically, let  $\tau(x, y)$  be the Kendall's Tau (rank correlation) of random variables  $x$  and  $y$ , such that, by definition,

$$\tau(x, y) = \text{Prob}((x - \tilde{x})(y - \tilde{y}) > 0) - \text{Prob}((x - \tilde{x})(y - \tilde{y}) < 0)$$

where  $(\tilde{x}, \tilde{y})$  is an independent copy of  $(x, y)$ . For the practically important class of elliptical distributions, the following characterization is possible.

**PROPOSITION 2** *Consider (8) with the recovery mapping functions  $C$  strictly increasing. Define  $Y_i = \mu_i + b_i \cdot Z + \xi_i$  and*

$$\sigma_i = \sqrt{b_i \cdot b_i + \sigma_{\xi_i}^2}, \quad i = 1, \dots, N$$

*If  $(X_1, \dots, X_N, Y_1, \dots, Y_N)$  belong to the class of continuous elliptical distributions, then*

$$\tau(X_i, X_j) = 2\pi^{-1} \sin^{-1}(a_i \cdot a_j), \quad i \neq j \quad (10a)$$

$$\tau(R_i, R_j) = \tau(Y_i, Y_j) 2\pi^{-1} \sin^{-1}(b_i \cdot b_j / (\sigma_i \sigma_j)), \quad i \neq j \quad (10b)$$

$$\tau(R_i, X_j) = \tau(Y_i, X_j) 2\pi^{-1} \sin^{-1}(b_i \cdot a_j / \sigma_i) \quad (10c)$$

**PROOF** All three statements are derived the same way, so we just concentrate on (10c). First, notice that the normalization assumption on the  $X_i$  implies a linear correlation coefficient between  $Y_i$  and  $X_j$  of just  $b_i \cdot a_j / \sigma_i$  ( $i \neq j$ ). The equality

$$\tau(Y_i, X_j) = 2\pi^{-1} \sin^{-1}(b_i \cdot a_j / \sigma_i)$$

is a consequence of a general result for elliptical distributions in Lindskog *et al* (2003); the equation  $\tau(R_i, X_j) = \tau(Y_i, X_j)$  follows from the fact that Kendall's Tau is invariant under strictly increasing transformations (see, eg, Embrechts *et al*, 2001).

### 3.1 Portfolio loss distribution

The computation of the portfolio loss distribution in this model is more involved than for the Gaussian copula model. This is because the conditional loss distributions of individual obligors are in general – due to the idiosyncratic randomness of recovery – not just two-point distributions. However, since losses from individual obligors are independent conditional on the factors, we can still compute the conditional portfolio loss distribution as the convolution product of all of the single-name loss distributions.

To do this we assume that a positive loss unit  $u$  is given such that we can approximate the distribution of each  $l_i$  by the discrete distribution of the integer valued  $k_i$ ,  $l_i = k_i u$ . Since  $l_i \in [0, l_i^{\max}]$  the support of the discrete distribution must lie on the non-negative integers up to some finite value  $k_i^{\max}$  ( $\approx l_i^{\max}/u$ , but depending on the details of the chosen discretization). What we need in building the portfolio loss distribution is a discretization of the quantity  $\ell_i := l_i 1_{\tau_i \leq T}$ , which describes the distribution of losses due to default before time  $T$ . Since recovery and default are independent when we condition on  $Z$ , we have

$$\begin{aligned} \text{Prob}(\ell_i \leq x | Z) &= \text{Prob}(l_i \leq x | Z) \cdot \text{Prob}(\tau_i \leq T | Z) + \text{Prob}(\tau_i > T | Z) \\ &= 1 - p_i(Z) (1 - \text{Prob}(l_i \leq x | Z)) \end{aligned}$$

and we can therefore get a discrete distribution for  $\ell_i$ , denoted  $P^{(i)}(\cdot | Z)$ , from the discrete distribution for  $l_i$ . The support of  $P^{(i)}(\cdot | Z)$  lies on the non-negative integers up to  $k_i^{\max}$ .

Now let  $L$  denote the total portfolio loss (measured in loss units) and let ‘ $\wedge$ ’ denote the discrete convolution product over the integers. Then the conditional portfolio loss distribution is given by

$$\text{Prob}(L = k | Z) = \left( \bigwedge_{i=1}^N P^{(i)}(\cdot | Z) \right) (k)$$

where we note that the convolution product is well-defined since all factors have finite support. Clearly, the conditional portfolio loss distribution also has finite support. In the computation of the convolution product we can make use of standard Fourier transform techniques, but we may also do a recursive computation. Specifically, let  $L_n$ ,  $1 \leq n < N$ , be the loss (measured in loss units) in the sub-portfolio consisting of  $n$  obligors. Then we have the following recursive relation between the conditional distributions of  $L_n$  and  $L_{n+1}$ :

$$\text{Prob}(L_{n+1} = K | Z) = \sum_{k=0}^{k_{n+1}^{\max}} \text{Prob}(L_n = K - k | Z) P^{(n+1)}(k) \quad (11)$$

We may use this recursion to compute the conditional loss distribution of the portfolio from the boundary case of the empty portfolio for which we have  $\text{Prob}(L_0 = K | Z) = \delta_{K,0}$ .

Partial derivatives may be computed from (11), but in general the direct recursive solution of (11) for the loss distribution of the portfolio with one obligor removed is numerically unstable. This problem is of a purely numerical nature and may be overcome, eg, by the use of Fourier transform or matrix inversion techniques. Note that the problem exists only for the general case where recovery has an idiosyncratic component.

### 3.2 Specific model: cumulative Gaussian recovery

We consider the specific model defined from (8) by letting  $Z$ ,  $\varepsilon_i$  and  $\xi_i$  be Gaussian and by assuming that the mapping functions  $C_i(\cdot)$  are all given by the standard cumulative Gaussian distribution:  $C_i = \Phi$ , for all  $i$ .

For later use, and to further characterize the model, we now list some useful results for losses and conditional defaults. First note that the  $X_i$ s are also standard Gaussian and that

$$p_i(Z) = \Phi \left( \frac{\Phi^{-1}(p_i) - a_i \cdot Z}{\sqrt{1 - \|a_i\|^2}} \right), \quad i = 1, \dots, N \quad (12)$$

as a special case of Proposition 1.

PROPOSITION 3 *Define*

$$\sigma_i = \sqrt{b_i \cdot b_i + \sigma_{\xi_i}^2}$$

and  $\rho_i = a_i \cdot b_i / \sigma_i$  and let  $\varphi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$  be the standard Gaussian density. Also, let  $\Phi_2(\cdot, \cdot; \rho)$  be the bivariate cumulative Gaussian distribution function with correlation  $\rho$ . Then we have

$$\text{Prob}(R_i \leq x) = \Phi \left( \frac{\Phi^{-1}(x) - \mu_i}{\sigma_i} \right) \quad (13a)$$

$$\text{Prob}(R_i \in [x, x + dx]) = dx \cdot \varphi \left( \frac{\Phi^{-1}(x) - \mu_i}{\sigma_i} \right) \left( \varphi(\Phi^{-1}(x)) \sigma_i \right)^{-1} \quad (13b)$$

$$E(R_i) = \Phi \left( \frac{\mu_i}{\sqrt{1 + \sigma_i^2}} \right) \quad (13c)$$

$$V(R_i) = \Phi_2 \left( \frac{\mu_i}{\sqrt{1 + \sigma_i^2}}, \frac{\mu_i}{\sqrt{1 + \sigma_i^2}}, \frac{\sigma_i^2}{1 + \sigma_i^2} \right) - \Phi \left( \frac{\mu_i}{\sqrt{1 + \sigma_i^2}} \right)^2 \quad (13d)$$

$$l_i(Z) = l_i^{\max} \left( 1 - \Phi \left( \frac{\mu_i + b_i \cdot Z}{\sqrt{1 + \sigma_{\xi_i}^2}} \right) \right) \tag{13e}$$

$$\text{Prob}(R_i \leq x | Z) = \Phi \left( \frac{\Phi^{-1}(x) - \mu_i - \beta_i \cdot Z}{\sigma_{\xi_i}} \right) \tag{13f}$$

$$E(R_i | \tau_i \leq T) = p_i^{-1} \Phi_2 \left( \frac{\mu_i}{\sqrt{1 + \sigma_i^2}}, c_i; \frac{-\rho_i \sigma_i}{\sqrt{1 + \sigma_i^2}} \right) \tag{13g}$$

PROOF Introduce  $Y_i = \mu_i + b_i \cdot Z + \xi_i$ , which is obviously Gaussian with mean  $\mu_i$  and standard deviation  $\sigma_i$ . The statement (13a) follows from

$$\begin{aligned} \text{Prob}(R_i \leq x) &= \text{Prob}(Y_i \leq \Phi^{-1}(x)) = \text{Prob} \left( \frac{Y_i - \mu_i}{\sigma_i} \leq \frac{\Phi^{-1}(x) - \mu_i}{\sigma_i} \right) \\ &= \Phi \left( \frac{\Phi(x) - \mu_i}{\sigma_i} \right) \end{aligned}$$

Differentiation of this expression results in (13b). As for statements (13c) and (13d), we note that

$$\begin{aligned} E(R_i) &= \int_{-\infty}^{\infty} \Phi(\mu_i + \sigma_i x) \varphi(x) dx \\ V(R_i) &= \int_{-\infty}^{\infty} \Phi(\mu_i + \sigma_i x)^2 \varphi(x) dx - E(R_i)^2 \end{aligned}$$

Expressions for these integrals can be found in the Appendix, yielding the stated results. Statement (13e) follows directly from Proposition 1, and statement (13f) follows from (2) and (13c), after observing that  $Y_i | Z$  is Gaussian with mean  $\mu_i + b_i \cdot Z$  and variance  $\sigma_{\xi_i}^2$ . To prove (13g) note that the correlation between  $Y_i$  and  $X_i$  is  $\rho_i$  and that, conditional on  $X_i = x$ , the distribution of  $Y_i$  is

$$N(\mu_i + \rho_i \sigma_i x, \sigma_i^2 \sqrt{1 - \rho_i^2})$$

It then follows easily from (13c) that

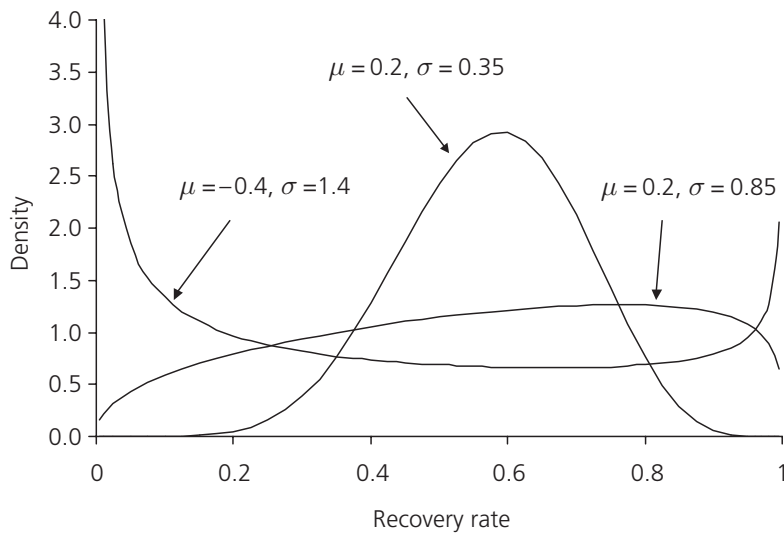
$$E(R_i | X_i = x) = \Phi \left( \frac{\mu_i + \rho_i \sigma_i x}{\sqrt{1 + \sigma_i^2 (1 - \rho_i^2)}} \right)$$

whereby

$$E(R_i | \tau_i \leq T) = E(R_i | X_i \leq c_i) = p_i^{-1} \int_{-\infty}^{c_i} \Phi \left( \frac{\mu_i + \rho_i \sigma_i x}{\sqrt{1 + \sigma_i^2 (1 - \rho_i^2)}} \right) \varphi(x) dx$$

Using the result (30c) in the Appendix proves the proposition. □

**FIGURE 3** Recovery rate densities in cumulative Gaussian model.



The densities are given by formula (13b) in Proposition 3. In the graph,  $\sigma = \sigma_i = \sqrt{\|b\|^2 + \sigma_\xi^2}$  (and we have suppressed the obligor index  $i$ ).

Calibration of recovery model parameters could be done by matching moments and rank correlation properties of the recovery distribution to empirical data or to a proprietary view. The results in Propositions 2 and 3 are useful for this. Importantly, in any calibration scheme we must ensure that the model remains consistent with the default swap market. For this we use (13g) in conjunction with (3) and (4). We note that if, for instance, recovery on default swaps (ie, the expectation of recovery conditioned on default) is assumed to be time-independent, then it follows from (13g) that one or more of the parameters  $\sigma$ ,  $\mu$  or  $c$  must in general be time-dependent. Alternatively, if the parameters are assumed to be time-independent, the recovery expectation must be time-dependent. In either case it is straightforward to fit the parameters in a bootstrap procedure.

The specific model for random recovery just described is tractable and capable of producing a variety of distributions of recovery rates. The use of a cumulative Gaussian distribution may be unconventional, but we note that the more traditional choice of a specification in terms of the beta-distribution leads to a less tractable model. Furthermore, the range of possible recovery densities does not seem to differ much between the two specifications,<sup>9</sup> so any empirical support for using beta-distributions (see, eg, Gupton and Stein, 2002) carries over to the cumulative Gaussian specification. For illustration, Figure 3 graphs the (unconditional) recovery densities for various parameter specifications for our model. Finally, we note that extensions to a Student’s  $t$  copula are straightforward.

<sup>9</sup> In fact, when the first two recovery rate moments are set to fit the moments of a given beta distribution, the model density is nearly indistinguishable from the beta density.

### 3.3 Large portfolio limit

To gain intuition, we shall now consider some useful limit results arising when the size of the underlying portfolio becomes large, in the sense that  $N \rightarrow \infty$ . Such results are convenient and, as documented in Schönbucher (2001) and Vasicek (2003), large-portfolio limit loss distributions are often remarkably accurate approximations for finite-sized portfolios, especially in the upper tail. Moreover, for empirical estimations done on economy-wide data, the large-portfolio approximation is often natural to consider.

To ensure the existence of a large-portfolio limit, we shall need to impose some structure on the portfolio:

ASSUMPTION 1 *There exist systematic factors  $Z$  and a function  $h$  such that (point-wise)*

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^N p_i(Z) l_i(Z) = h(Z) \tag{14}$$

Moreover, the composition of the portfolio satisfies

$$\lim_{N \rightarrow \infty} N^{-2} \sum_{i=1}^N (l_i^{\max})^2 = 0 \tag{15}$$

PROPOSITION 4 *Let the distribution function for the (independent) systematic factors  $Z$  be denoted  $F^Z$ . Under the terms of Assumption 1, we have*

$$\lim_{N \rightarrow \infty} \text{Prob}(L/N \leq y) = \text{Prob}(h(Z) \leq y) = \int_{h(z) \leq y} dF^Z(z)$$

PROOF First notice that

$$\begin{aligned} E(L/N | Z) &= E \left( N^{-1} \sum_{i=1}^N l_i 1_{\tau_i < T} \mid Z \right) \\ &= N^{-1} \sum_{i=1}^N E(l_i | Z) E(1_{\tau_i < T} | Z) = N^{-1} \sum_{i=1}^N p_i(Z) l_i(Z) \end{aligned}$$

where we have relied on conditional independence in the second equality. Conditional on  $Z$ , the variance of the average loss  $L/N$  is bounded by  $N^{-2} \sum_i (l_i^{\max})^2$  which, by assumption, approaches 0 for large  $N$ . We can use Chebychev’s inequality to show that the random variable  $L/N | Z$  converges in probability to  $\lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^N p_i(Z) l_i(Z) = h(Z)$ . This, in turn, implies convergence in distribution:

$$\lim_{N \rightarrow \infty} \text{Prob}(L/N \leq y | Z) = 1_{h(Z) \leq y}$$

Now,

$$\begin{aligned} \lim_{N \rightarrow \infty} \text{Prob}(L/N \leq y) &= \lim_{N \rightarrow \infty} \text{E}(\text{Prob}(L/N \leq y | Z)) \\ &= \text{E}\left(\lim_{N \rightarrow \infty} \text{Prob}(L/N \leq y | Z)\right) = \text{E}\left(\mathbf{1}_{h(Z) \leq y}\right) \end{aligned}$$

where interchange of limit and expectation is justified by the fact that  $\text{Prob}(L/N \leq y | Z)$  is bounded.  $\square$

The conditions in Assumption 1 are quite weak. For instance, (15) is satisfied by any portfolio where the  $l^{\max}$  are bounded for all  $i$ . Equation (14) is trivially satisfied by homogeneous portfolios where  $p_i(Z) \equiv p(Z)$  and  $l_i(Z) \equiv l(Z)$  are independent of  $i$ ; in this case, obviously  $h(Z) = p(Z)l(Z)$ .

For the popular special case of a one-dimensional systematic factor structure, the result in Proposition 4 simplifies considerably. Specifically, assuming that the conditional loss-weighted default probability function  $h : \mathbb{R} \rightarrow \mathbb{R}$  is strictly decreasing in its argument, we just get

$$\begin{aligned} \lim_{N \rightarrow \infty} \text{Prob}(L/N \geq y) &= \text{Prob}(h(Z) \geq y) \\ &= \text{Prob}(Z \leq h^{-1}(y)) = F^Z(h^{-1}(y)) \end{aligned} \quad (16)$$

We note that the assumption that  $h(z)$  is decreasing in  $z$  is natural and consistent with our earlier discussion of loss-given-default being positively related to portfolio default frequency. Indeed, interpreting  $Z$  as a proxy for the state of the economic environment, it is reasonable to assume that all  $p_i(Z)$ s and  $l_i(Z)$ s are adversely affected (that is, increase) by a low outcome of  $Z$ .

For a concrete model parameterization, consider the following proposition, valid for a homogeneous portfolio (so we omit obligor indices on parameters):

**PROPOSITION 5** *Consider an homogeneous portfolio in a one-factor version of the model in Section 3.2, with the identical default probability of all obligors denoted  $\bar{p}$ . We have*

$$\lim_{N \rightarrow \infty} \text{Prob}(L/N \leq y) = \Phi(-h^{-1}(y))$$

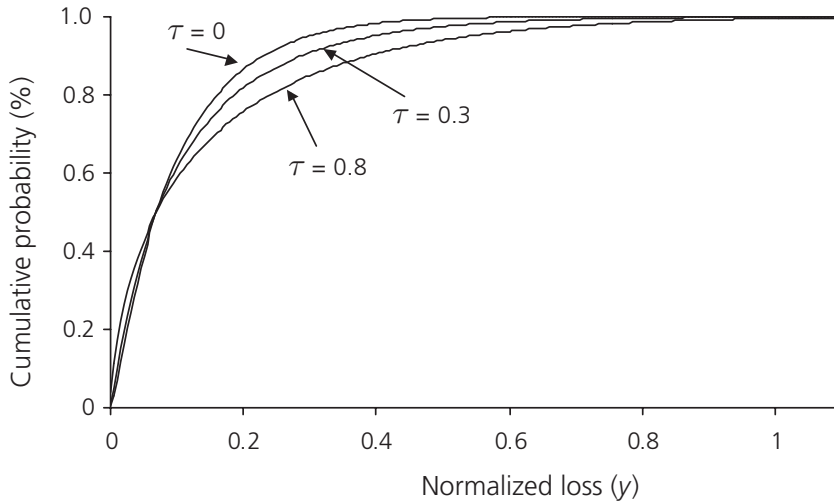
$$h(z)/l^{\max} = \Phi\left(\frac{\Phi^{-1}(\bar{p}) - az}{\sqrt{1 - a^2}}\right) \left(1 - \Phi\left(\frac{\mu + bz}{\sqrt{1 + \sigma_\xi^2}}\right)\right)$$

for constants  $a, \mu, b, \sigma_\xi$ , where  $0 < a < 1$  and  $b \geq 0$ .

**PROOF** The function  $h$  is defined in Assumption 1 and can be computed explicitly by combining equation (12) and (13e). For the given restrictions on  $a$  and  $b$ , it is easy to see that  $h$  is strictly decreasing. As the model in Section 3.2 has standard Gaussian systematic factors, the proposition then follows directly from (16).  $\square$

The result in Proposition 5 is straightforward to implement and gives an attractive alternative to the constant-recovery result in Vasicek (2003) (which is the special

**FIGURE 4** Large-portfolio distribution, cumulative Gaussian recovery model.



Homogeneous portfolio in one-factor Gaussian copula, with cumulative Gaussian recovery. The graph shows  $\text{Prob}([N]^{\max}(1 - E(R))^{-1}L \leq y)$ . Model parameters are set such that  $E(R) = 0.5$  (ie,  $\mu = 0$ ) and  $\rho(X_i, X_j) = 0.25$  (ie,  $a = 0.25$ ). Parameters  $b$  and  $\sigma_\xi$  are determined such that  $V(R) = 1/15$  and Kendall's Tau,  $\tau(X_i, R_i)$ , is as listed on the graphs.

case of Proposition 5, for  $b = 0$ ). Determination of the parameters of the recovery model ( $a, \mu, b, \sigma_\xi$ ) can be done straightforwardly by matching moments and rank correlation, using the expressions in Propositions 1 and 2. For illustration, in Figure 4, we have locked the values of  $E(R)$ ,  $V(R)$ , and  $\rho(X_i, X_j) = a^2$  and consider the effect on the large portfolio limit distribution of varying the recovery/default rank correlation  $\tau(R_i, X_i) = 2\pi^{-1}\sin^{-1}(ba/\sigma)$ ,  $\sigma^2 \equiv \sigma_\xi^2 + b^2$ . As one would expect, increasing the rank correlation will increase the likelihood of having high-loss scenarios, making the upper distribution tail significantly fatter.

To support empirical estimation, we round off this section with a few results on the rank correlation between default frequency and average recovery. Let  $\Delta_N := \sum_{i=1}^N 1_{\tau_i \leq T}$  be the total number of defaults and define the default frequency  $f_N := \Delta_N/N$  and average recovery

$$r_N := \begin{cases} L/\Delta_N, & \Delta_N > 0 \\ 0, & \Delta_N = 0 \end{cases}$$

**PROPOSITION 6** Consider a homogeneous portfolio with non-zero default probability and let  $p_i(Z) \equiv p(Z)$  and  $l_i(Z) \equiv l(Z)$  for all  $i$ . Then

$$\tau(f_\infty, r_\infty) = \tau(p(Z), l(Z))$$

If the factor setting of equation (8) holds and: (a) systematic factors  $Z$  are jointly elliptical; (b) the recovery mapping function  $C$  is strictly increasing; and (c) the distribution function of residuals  $\varepsilon_i$  is strictly increasing; then

$$\tau(f_\infty, l_\infty) = \tau(a \cdot Z, b \cdot Z) = 2\pi^{-1} \sin^{-1} \left( b \cdot a / \sqrt{\|a\|^2 + \|b\|^2} \right)$$

PROOF Notice that  $E(f_N | Z) = N^{-1} \sum_{i=1}^N E(1_{\tau_i \leq T} | Z) = N^{-1} N p(Z) = p(Z)$ . As the variance of  $f_N | Z$  is bounded from above by  $N^{-1}$ ,  $f_N$  converges to the random variable  $p(Z)$  in probability. Assuming that at least one default takes place, we notice that

$$\begin{aligned} E(r_N | Z) &= E \left( \sum_{i=1}^N 1_{\tau_i \leq T} l_i / \Delta_N \middle| Z \right) \\ &= E \left( E \left( \sum_{i=1}^N 1_{\tau_i \leq T} l_i / \Delta_N \middle| Z, 1_{\tau_1 \leq T}, \dots, 1_{\tau_N \leq T} \right) \right) \end{aligned}$$

where the outer expectation in the last expression is with respect to the  $N$  default indicator functions. Due to conditional independence, we have

$$\begin{aligned} E \left( \sum_{i=1}^N 1_{\tau_i \leq T} l_i / \Delta_N \middle| Z, 1_{\tau_1 \leq T}, \dots, 1_{\tau_N \leq T} \right) &= \sum_{i=1}^N 1_{\tau_i \leq T} E(l_i | Z) / \Delta_N \\ &= l(Z) \end{aligned}$$

As the probability of at least one default taking place approaches 1 for  $N \rightarrow \infty$ , and the variance of  $f_N | Z$  is bounded from above by  $(l^{\max})^2 N^{-1}$ ,  $r_N$  converges to the random variable  $l(Z)$  in probability. The first part of the proposition then follows. To prove the second part notice that (c) and Proposition 1 ensures that  $p(Z)$  is strictly decreasing in  $a \cdot Z$ . Similarly, (b) ensures that  $l(Z)$  is strictly decreasing in  $b \cdot Z$ . By the invariance of Kendall's Tau under strictly increasing transformations, we get that  $\tau(f_\infty, l_\infty) = \tau(-a \cdot Z, -b \cdot Z) = \tau(a \cdot Z, b \cdot Z)$ , where the second equality follows from the definition of Kendall's Tau. The statement

$$\tau(a \cdot Z, b \cdot Z) = 2\pi^{-1} \sin^{-1} \left( b \cdot a / \sqrt{\|a\|^2 + \|b\|^2} \right)$$

follows from the elliptical distribution of  $Z$  and the proof of Proposition 2.  $\square$

#### 4 RANDOM FACTOR LOADINGS

In this section we consider the special case of the general framework of Section 2 where the factor loadings depend deterministically on the factors. More specifically, let  $Z_j, j = 1, \dots, d$ , and  $\varepsilon_i, i = 1, \dots, N$ , be independent random variables with zero means and unit variances. Then we consider

$$\left. \begin{aligned} X_i &= a_i(Z) \cdot Z + \nu_i \varepsilon_i + m_i \\ l_i &= l_i^{\max} (1 - R_i) \end{aligned} \right\} i = 1, \dots, N \quad (17)$$

where  $R_i \in [0, 1]$  are fixed,

$$\nu_i := \sqrt{1 - \mathbb{V}[a_i(Z) \cdot Z]}$$

and  $m_i := -\mathbb{E}[a_i(Z) \cdot Z]$ , such that  $X_i$  has zero mean and unit variance. Note that

$$\nu_i = \sqrt{1 - \int_{\mathbb{R}^d} (a_i(z) \cdot z)^2 dF^Z(z) + m_i^2} \tag{18}$$

$$m_i = - \int_{\mathbb{R}^d} a_i(z) \cdot z dF^Z(z) \tag{19}$$

where  $F^Z$  is the distribution function for  $Z$ . For this model we can build the conditional loss distribution by the simple recursion (6) used for the Gaussian copula model. The requisite conditional default probabilities are given by

$$p_i(Z) = F_i^\varepsilon \left( \frac{c_i - a_i(Z) \cdot Z - m_i}{\nu_i} \right) \tag{20}$$

where  $F_i^\varepsilon$  is the cumulative distribution of  $\varepsilon_i$ . The computation of partial derivatives closely follows the Gaussian copula model as described in Section 2.1. In the model (17) the probability of default before some horizon  $T$  is

$$\begin{aligned} p_i &= \text{Prob}(\tau_i \leq T) = \text{Prob}(X_i \leq c_i) = \mathbb{E} \left[ \text{Prob} \left( \varepsilon_i \leq \frac{c_i - a_i(Z) \cdot Z - m_i}{\nu_i} \mid Z \right) \right] \\ &= \int_{\mathbb{R}^d} F_i^\varepsilon \left( \frac{c_i - a_i(z) \cdot z - m_i}{\nu_i} \right) dF^Z(z) \end{aligned} \tag{21}$$

This expression – computed either numerically or, if possible, by a closed-form solution – allows us to calibrate the model by determining the trigger levels  $c_i$  from default probabilities.

In practice, it is advantageous to work with a “separable” structure

$$a_i(Z) = (a_{i,1}(Z_1), \dots, a_{i,d}(Z_d)) \tag{22}$$

which will simplify variance and mean computations. For instance,

$$\mathbb{V}(a_i(Z) \cdot Z) = \sum_{j=1}^d \mathbb{V}(a_{ij}(Z_j) Z_j) \tag{23}$$

$$\text{Cov}(a_i(Z) \cdot Z, a_j(Z) \cdot Z) = \sum_{k=1}^d \text{Cov}(a_{ik}(Z_k) Z_k, a_{jk}(Z_k) Z_k) \tag{24}$$

and so forth.

#### 4.1 Specific model: two-point loadings distribution with Gaussian factors

Above we defined a fairly rich class of random factor loading models differing in the functional relationship between factors and loadings and in the choice of distributions for the factors and residuals.

For illustration, we shall now study in more detail a simple model, building on the standard Gaussian copula specification. Specifically, in this model  $Z_j$ ,  $j = 1, \dots, d$ , and  $\varepsilon_i$ ,  $i = 1, \dots, N$ , are standard Gaussian variates and the factor loadings are given by a *two-point distribution*:

$$a_{ij}(Z_j) = \begin{cases} \alpha_{ij}, & Z_j \leq \theta_{ij} \\ \beta_{ij}, & Z_j > \theta_{ij} \end{cases} \quad (25)$$

where  $\alpha_{ij}$ ,  $\beta_{ij}$  are positive constants and  $\theta_{ij} \in \mathbb{R}$ . Note that we have assumed the separable structure of equation (22). Loosely, we can think of this as a regime-switching model where loading  $j$  takes value  $\alpha_{ij}$  with probability  $\Phi(\theta_{ij})$  and value  $\beta_{ij}$  with probability  $1 - \Phi(\theta_{ij})$ . If  $\alpha_{ij} > \beta_{ij}$  then the factor loadings decrease in  $Z_j$  and thus, intuitively, asset values couple more strongly to “the economy” in bad times than in good. Clearly, the special case of constant, factor-independent  $a_{ij}$ s is the Gaussian copula model, but in general the default drivers,  $X_i$  (and thus the copula), will *not* be Gaussian.

While analytical tractability is not particularly important<sup>10</sup> we note that more general, yet still tractable, specifications than (13) exist. For instance, extending the two-point specification to an  $N$ -point specification is trivial and allows one to approximate a given smooth function to arbitrary precision. A piecewise linear (rather than piecewise flat) factor loading function is also tractable. Finally, we note that (25) is a special case of the smooth and analytically tractable specification

$$\begin{aligned} a_i(\mathbf{Z}) &= \left( \chi_{i,1} + \gamma_{i,1} \Phi(\mu_{i,1} Z_1 + \nu_{i,1}), \dots, \chi_{i,d} + \gamma_{i,d} \Phi(\mu_{i,d} Z_d + \nu_{i,d}) \right) \\ &= \chi_i + \text{diag}(\gamma_i) \Phi(\mu_i \cdot \mathbf{Z} + \nu_i) \end{aligned} \quad (26)$$

where we introduced the  $d$ -dimensional vectors  $\chi_i \geq 0$ ,  $0 \leq \gamma_i \leq 1 - \chi_i$ ,  $\mu_i$ , and  $\nu_i$ . In the last equation, we understand that  $\Phi(\cdot)$  is to be applied componentwise. We note that  $0 \leq a_i(\mathbf{Z}) \leq \chi_i + \gamma_i$ , and must thus require that

$$\|\chi_i + \gamma_i\|^2 \leq 1$$

We would typically expect  $\mu_i$  to be negative, such that factor loadings increase when the factors decrease. Although the specification (26) is tractable it requires additional parameters and involves fairly lengthy expressions. For the purposes of this paper the simpler specification (25) suffices and will serve to illustrate the basic idea of random factor loadings. Moment properties of (25) are straightforward and are listed below.

<sup>10</sup>Non-tractable factor loading functions can be handled by numerical quadrature coupled with some type of function caching.

PROPOSITION 7 For the model in (25), we have

$$m_i = -\sum_{j=1}^d \left( -\alpha_{ij}\varphi(\theta_{ij}) + \beta_{ij}\varphi(\theta_{ij}) \right)$$

$$v_i = \sqrt{1 - \sum_{j=1}^d V_{ij}}$$

where

$$V_{ij} = \alpha_{ij}^2 \left( \Phi(\theta_{ij}) - \theta_{ij}\varphi(\theta_{ij}) \right) + \beta_{ij}^2 \left( \theta_{ij}\varphi(\theta_{ij}) + 1 - \Phi(\theta_{ij}) \right) - \left( -\alpha_{ij}\varphi(\theta_{ij}) + \beta_{ij}\varphi(\theta_{ij}) \right)^2$$

PROOF Using the results of Lemma 5, we immediately get

$$\begin{aligned} E(a_{ij}(Z_j)Z_j) &= E\left(\alpha_{ij}1_{Z_j \leq \theta_{ij}}Z_j + \beta_{ij}1_{Z_j > \theta_{ij}}Z_j\right) = -\alpha_{ij}\varphi(\theta_{ij}) + \beta_{ij}\varphi(\theta_{ij}) \\ E(a_{ij}(Z_j)^2Z_j^2) &= E\left(\alpha_{ij}^21_{Z_j \leq \theta_{ij}}Z_j^2 + \beta_{ij}^21_{Z_j > \theta_{ij}}Z_j^2\right) \\ &= \alpha_{ij}^2\left(\Phi(\theta_{ij}) - \theta_{ij}\varphi(\theta_{ij})\right) + \beta_{ij}^2\left(\theta_{ij}\varphi(\theta_{ij}) + (1 - \Phi(\theta_{ij}))\right) \end{aligned}$$

From this  $V(a_{ij}(Z_j)^2Z_j^2) = V_{ij}$ , with  $V_{ij}$  given above. The result then follows from equation (23).  $\square$

To characterize the dependence structure of the model we give the following result.

PROPOSITION 8 Define  $\bar{\theta}_{ijk} = \max(\theta_{ik}, \theta_{jk})$  and  $\underline{\theta}_{ijk} = \min(\theta_{ik}, \theta_{jk})$ . Then

$$E(a_{ij}) = \alpha_{ij}\Phi(\theta_{ij}) + \beta_{ij}(1 - \Phi(\theta_{ij})) \tag{27a}$$

$$V(a_{ij}) = \alpha_{ij}^2\Phi(\theta_{ij}) + \beta_{ij}^2(1 - \Phi(\theta_{ij})) - E(a_{ij})^2 \tag{27b}$$

$$\rho(X_i, X_j) = \sum_{k=1}^d (E_{ijk} - m_i m_j), \quad i \neq j \tag{27c}$$

where

$$\begin{aligned} E_{ijk} &:= E(a_{ik}(Z_k)a_{jk}(Z_k)Z_k^2) \\ &= \alpha_{ik}\alpha_{jk}\left(\Phi(\underline{\theta}_{ijk}) - \underline{\theta}_{ijk}\varphi(\underline{\theta}_{ijk})\right) \\ &\quad + \beta_{ik}\beta_{jk}\left(1 - \Phi(\bar{\theta}_{ijk}) + \bar{\theta}_{ijk}\varphi(\bar{\theta}_{ijk})\right) \\ &\quad + \alpha_{ik}\beta_{jk}1_{\theta_{ik} \geq \theta_{jk}}\left(\Phi(\theta_{ik}) - \Phi(\theta_{jk}) + \theta_{jk}\varphi(\theta_{jk}) - \theta_{ik}\varphi(\theta_{ik})\right) \\ &\quad + \beta_{ik}\alpha_{jk}1_{\theta_{jk} \geq \theta_{ik}}\left(\Phi(\theta_{jk}) - \Phi(\theta_{ik}) + \theta_{ik}\varphi(\theta_{ik}) - \theta_{jk}\varphi(\theta_{jk})\right) \end{aligned}$$

PROOF The first equation is obvious. The second follows from

$$\begin{aligned}
 & \mathbb{E} \left( \left( \alpha_{ij} 1_{Z_j \leq \theta_{ij}} + \beta_{ij} 1_{Z_j > \theta_{ij}} \right)^2 \right) \\
 &= \mathbb{E} \left( \alpha_{ij}^2 1_{Z_j \leq \theta_{ij}} + \beta_{ij}^2 1_{Z_j > \theta_{ij}} + 2\alpha_{ij}\beta_{ij} 1_{Z_j > \theta_{ij}} 1_{Z_j \leq \theta_{ij}} \right) \\
 &= \mathbb{E} \left( \alpha_{ij}^2 1_{Z_j \leq \theta_{ij}} + \beta_{ij}^2 1_{Z_j > \theta_{ij}} \right) \\
 &= \alpha_{ij}^2 \Phi(\theta_{ij}) + \beta_{ij}^2 (1 - \Phi(\theta_{ij}))
 \end{aligned}$$

To prove the third equation we notice that

$$\begin{aligned}
 & \mathbb{E} \left( a_{ik}(Z_k) a_{jk}(Z_k) Z_k^2 \right) \\
 &= \mathbb{E} \left( \alpha_{ik} \alpha_{jk} 1_{Z_k \leq \theta_{ik}} 1_{Z_k \leq \theta_{jk}} Z_k^2 + \beta_{ik} \beta_{jk} 1_{Z_k > \theta_{ik}} 1_{Z_k > \theta_{jk}} Z_k^2 \right) \\
 &\quad + \mathbb{E} \left( \alpha_{ik} \beta_{jk} 1_{Z_k \leq \theta_{ik}} 1_{Z_k > \theta_{jk}} Z_k^2 + \beta_{ik} \alpha_{jk} 1_{Z_k > \theta_{ik}} 1_{Z_k \leq \theta_{jk}} Z_k^2 \right) \\
 &= \mathbb{E} \left( \alpha_{ik} \alpha_{jk} 1_{Z_k \leq \min(\theta_{ik}, \theta_{jk})} Z_k^2 + \beta_{ik} \beta_{jk} 1_{Z_k > \max(\theta_{ik}, \theta_{jk})} Z_k^2 \right) \\
 &\quad + \mathbb{E} \left( \alpha_{ik} \beta_{jk} 1_{\theta_{jk} < Z_k \leq \theta_{ik}} Z_k^2 + \beta_{ik} \alpha_{jk} 1_{\theta_{ik} < Z_k \leq \theta_{jk}} Z_k^2 \right)
 \end{aligned}$$

From Lemma 5 in the Appendix, the value of this expression is  $E_{ijk}$ , as given in the proposition. The final result now follows from

$$\rho(X_i, X_j) = \mathbb{E}(X_i X_j) = \text{cov}(a_i(Z) \cdot Z, a_j(Z) \cdot Z)$$

and

$$\begin{aligned}
 & \text{cov}(a_i(Z) \cdot Z, a_j(Z) \cdot Z) \\
 &= \sum_{k=1}^d \text{cov}(a_{ik}(Z_k) Z_k, a_{jk}(Z_k) Z_k) \quad \square
 \end{aligned}$$

If we restrict ourselves to low-dimensional models, the specification (25) is simple enough to allow for closed-form default probabilities. For instance:

PROPOSITION 9 *Assume that the dimension of  $Z$  is  $d = 1$ . Then*

$$\begin{aligned}
 \text{Prob}(\tau_i \leq T) &= \text{Prob}(X_i \leq c_i) \\
 &= \Phi_2 \left( \frac{c_i - m_i}{\sqrt{\nu_i^2 + \alpha_i^2}}, \theta_i; \frac{\alpha_i}{\sqrt{\nu_i^2 + \alpha_i^2}} \right) + \Phi \left( \frac{c_i - m_i}{\sqrt{\nu_i^2 + \beta_i^2}} \right) \\
 &\quad - \Phi_2 \left( \frac{c_i - m_i}{\sqrt{\nu_i^2 + \beta_i^2}}, \theta_i; \frac{\beta_i}{\sqrt{\nu_i^2 + \beta_i^2}} \right) \tag{28}
 \end{aligned}$$

PROOF We have

$$\begin{aligned}
 &\text{Prob}(\tau_i \leq T) \\
 &= \text{Prob}(\alpha_i 1_{Z \leq \theta_i} Z + \beta_i 1_{Z > \theta_i} Z + \varepsilon_i \nu_i + m_i \leq c_i) \\
 &= \text{E} \left( \text{Prob} \left( \varepsilon_i \leq \frac{c_i - (\alpha_i 1_{Z \leq \theta_i} Z + \beta_i 1_{Z > \theta_i} Z) - m_i}{\nu_i} \middle| Z \right) \right) \\
 &= \text{E} \left( \Phi \left( \frac{c_i - (\alpha_i 1_{Z \leq \theta_i} Z + \beta_i 1_{Z > \theta_i} Z) - m_i}{\nu_i} \right) \right) \\
 &= \int_{-\infty}^{\theta_i} \Phi \left( \frac{c_i - \alpha_i Z - m_i}{\nu_i} \right) \varphi(Z) dZ + \int_{\theta_i}^{\infty} \Phi \left( \frac{c_i - \beta_i Z - m_i}{\nu_i} \right) \varphi(Z) dZ \tag{28a}
 \end{aligned}$$

and the proposition now follows from results in the Appendix. □

We note that the density of  $X_i$  can be obtained by differentiating (28) with respect to  $c_i$ .

### 4.2 Large-portfolio limits of two-point distribution model

With the specification (25), we are not only hoping to (crudely) mimic an empirical dependence of correlations on broad market conditions, we also hope to generate a base correlation skew when  $\alpha_{ij} > \beta_{ij}$ . To elaborate, consider the view of a senior tranche investor. This investor will only experience losses on his position when several names default together, that is, in scenarios where the systematic factors  $Z$  likely take on low values. If  $Z$  is low, however, the factor loading will be high, making it appear to the senior investor that correlations are quite high. For the equity investor, who is likely to experience losses even in scenarios where  $Z$  is not low, the effective factor loading will appear as a weighted average between  $\alpha_{ij}$  and  $\beta_{ij}$ . To the equity investor, the world will thus look as if correlations are of average magnitude. In fact, the convexity of tranche values in correlation will further reduce the “effective” correlation seen by the equity investor and increase that seen by the senior investor.

To lend some credence to the above intuitive argument, and to provide some useful results for large portfolio risk management under (25), we shall now briefly

examine the large-portfolio limit distribution of the model.<sup>11</sup> A direct examination of base correlation skews for a finite portfolio can be found in Section 6.2. To proceed, assume for simplicity that  $d = 1$  and that the portfolio is homogeneous.<sup>12</sup> Reflecting the homogeneity of the portfolio, we omit subscripts in the following result ( $\beta_{ij} = \beta$ ,  $\alpha_{ij} = \alpha$ ,  $m_i = m$ , and so forth).

**PROPOSITION 10** *Consider a homogeneous portfolio with a one-dimensional factor structure. Under (25), the following holds*

$$\lim_{N \rightarrow \infty} \text{Prob}(L/N \geq ly) = \Phi(\min(\Omega(y)/\alpha, \theta)) + 1_{\Omega/\beta > \theta} (\Phi(\Omega(y)/\beta) - \Phi(\theta))$$

$$\Omega(y) := c - \nu \Phi^{-1}(y) - m$$

**PROOF** Consider

$$\begin{aligned} h(Z) &= N^{-1} l^{-1} \text{E}(L|Z) = N^{-1} \text{E}\left(\sum_{i=1}^N 1_{\tau_i \leq T} \mid Z\right) \\ &= N^{-1} \sum_{i=1}^N \text{E}\left(1_{\tau_i \leq T} \mid Z\right) = \text{Prob}(X_1 \leq c \mid Z) \\ &= \text{Prob}\left(\varepsilon_1 \leq \frac{c - a(Z)Z - m}{\nu} \mid Z\right) = \Phi\left(\frac{c - 1_{Z \leq \theta} \alpha Z - 1_{Z > \theta} \beta Z - m}{\nu}\right) \end{aligned}$$

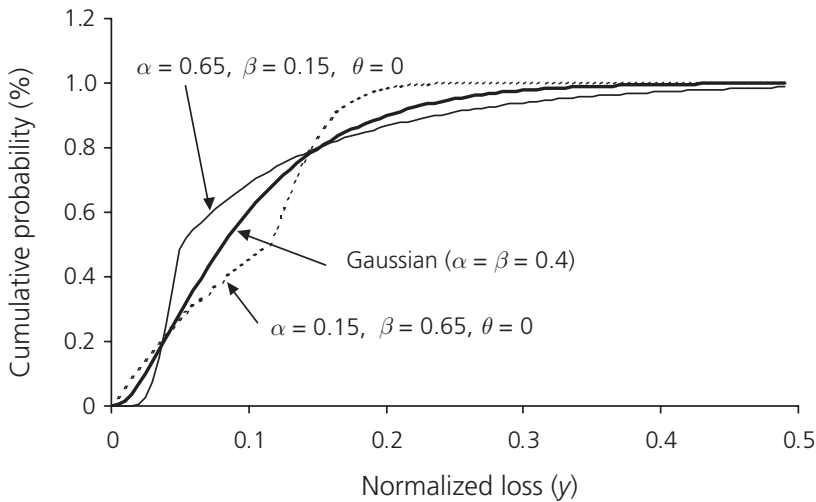
Then, by the usual diversification arguments,

$$\begin{aligned} &\lim_{N \rightarrow \infty} \text{Prob}(L/N \geq ly) \\ &= \text{Prob}(h(Z) \geq y) \\ &= \text{Prob}(a(Z)Z \leq \Omega(y)) \\ &= \text{Prob}(a(Z)Z \leq \Omega(y), Z \leq \theta) + \text{Prob}(a(Z)Z \leq \Omega(y), Z > \theta) \\ &= \text{Prob}(\alpha Z \leq \Omega(y), Z \leq \theta) + \text{Prob}(\beta Z \leq \Omega(y), Z > \theta) \\ &= \Phi(\min(\Omega(y)/\alpha, \theta)) + 1_{\Omega(y)/\beta > \theta} (\Phi(\Omega(y)/\beta) - \Phi(\theta)) \quad \square \end{aligned}$$

In Figure 5 below, we use the result in Proposition 10 to examine the loss distributions associated with different settings  $\beta$  and  $\alpha$ . We note that when  $\alpha > \beta$ , the loss distribution has qualitative properties consistent with the existence of a base skew: a fat upper tail, yet a reduced probability of generating small losses (cf Figure 5). (We should note that the somewhat non-smooth form of the large-portfolio loss

<sup>11</sup> Large-portfolio results for more general models than (25) are obviously possible, given the conditional independence of the model. The required procedure to obtain such results is similar to that given in the text.

<sup>12</sup> Extensions to non-homogeneous portfolios are trivial; see Section 3.3 for the required arguments and assumption.

**FIGURE 5** Large-portfolio distribution, two-point random factor loading model.

Homogenous portfolio in one-factor Gaussian copula, with two-point random factor loading model. The graphs shows  $\text{Prob}(L/(NI) < y)$ . The probability of default was 10% per name; parameters  $m$  and  $v$  were computed according to Proposition 7.

distribution in Figure 5 is caused by the simple two-point specification used and the absence of residual noise in the large-portfolio limit. In particular, we get a kink at a probability level of 0.5 which is the likelihood that the factor  $Z$  will cross the threshold  $\theta = 0$ ).

## 5 ALTERNATIVE CORRELATION SKEW MODELS

While the method presented in Section 4 is attractive for its simplicity and close relation to an established and well-understood framework, it is just one of many ways to generate a correlation skew. At the end of the day, any model that can generate a loss distribution similar to that of Figure 2 will produce a correlation skew. To briefly comment on a few alternative approaches, consider first an outright move to a copula different from the Gaussian.<sup>13</sup> For instance, among the better-known alternatives to the Gaussian copula, the Marshall-Olkin copula (see, eg, Giesecke, 2003, and references therein) can be verified to have enough flexibility to produce correlation skews. As the model apparently enjoys some popularity with practitioners, let us discuss it in more detail.

We first recall that the Marshall-Olkin is a Poisson-type model where the default intensity of credit  $i$ ,  $\lambda_i$ , is broken into an idiosyncratic component and a number of systematic components. Say that we have  $M$  systematic default shocks, modeled as independent Poisson processes  $Nk(t)$  with intensities  $\lambda k$ ,

<sup>13</sup>The random factor loading approach obviously makes the copula non-Gaussian, but the change of copula is, in a sense, implicit rather than explicit.

$k = 1, \dots, M$ ; these systematic shocks can potentially affect several firms simultaneously. To define whether a particular firm is sensitive to a particular systematic shock, one introduces zero- or one-valued indicator variables  $I_{ik}$  such that firm  $i$  will default with certainty the first time  $\bar{N}k(t)$  jumps if and only if  $I_{ik} = 1$ . If  $I_{ik} = 0$ , firm  $i$  is assumed completely insensitive to  $\bar{N}k(t)$ . Define also  $N$  idiosyncratic Poisson processes  $N_i^f(t)$ ,  $i = 1, \dots, N$  (“f” for “firm”), independent of each other and of the  $\bar{N}k(t)$ ,  $k = 1, \dots, M$ . Let the intensities of these Poisson processes be  $\lambda_i^f(t)$ ,  $i = 1, \dots, N$ . In the Marshall-Olkin copula model, we then express the aggregate default Poisson processes for the firms as

$$N_i(t) = \sum_{k=1}^M \bar{N}_k(t) I_{ik} + N_i^f(t), \quad i = 1, \dots, N$$

where the right-hand side is obviously Poisson (as a thinned sum of independent Poisson processes is again Poisson) with intensity

$$\lambda_i(t) = \sum_{k=1}^M \bar{\lambda}_k I_{ik} + \lambda_i^f(t), \quad i = 1, \dots, N$$

The dependence structure of the Marshall-Olkin model is determined by the matrix  $\{I_{ik}\}$  and the intensities  $\bar{\lambda}_k$ ,  $k = 1, \dots, M$ . By having a construct such as the  $\{I_{ik}\}$  matrix, the Marshall-Olkin copula is quite flexible and grants tight control of the relative probabilities of all possible combinations of simultaneous firm defaults. The resulting control over the total loss distribution in turn easily allows for the generation a correlation skew. On the other hand, the exact parameterization of the Marshall-Olkin copula is a rather formidable problem, given the abstract nature (and sheer number) of its parameters. We note that to make the model consistent across different CDOs, one really must calibrate a single matrix and a single set of intensities for all credits in the universe of traded credit default swaps. Such a calibration would likely be difficult to make robust, and strong assumptions will be needed to make it feasible. We also point out that even if a calibration method could be constructed, the model remains quite unwieldy and involve a number of non-trivial operational and computational issues, particularly in the computation of hedges. Similar issues arise with many other copulas and, to an even greater extent, the dynamic portfolio models of the reduced-form type discussed earlier in Section 1.

Stepping back to the Gaussian copula set-up, it is well-known that a Gaussian copula equipped with a non-constant correlation matrix will produce a non-flat base correlation skew. Some authors (see eg, Gregory and Laurent, 2004) have proposed correlation structures where correlations are high within certain “sectors” (arbitrary subportfolios) and low between sectors. However, when – as is perhaps natural – the sectors are taken to represent industrial or geographical categories, the resulting skew is far weaker than that observed in practice. One could imagine taking this approach one step further and attempting to “imply” a

correlation matrix that will reproduce the observed smile. This approach is perhaps best framed in a factor set-up, where we make factor loadings functions of the credit spreads. To be more specific, consider replacing equation (5) with<sup>14</sup>

$$X_i = g(s_i)Z + \varepsilon_i \sqrt{1 - \|g(s_i)\|^2}, \quad i = 1, \dots, N \quad (29)$$

where  $s_i$  represents the input credit spread of firm  $i$ ,  $i = 1, \dots, N$ , and  $g : \mathbb{R} \rightarrow [0, 1]^d$  is some mapping function. We stress that the  $g(s_i)$  are to be interpreted as constants, not as stochastic quantities;  $g(\cdot)$  can be “implied” (parametrically or non-parametrically) to best-fit the correlation skew. We note that to produce a skew consistent with that seen in the market,  $g$  needs to produce a correlation matrix such that the senior investor (who is affected more by low-spread credits than is the equity investor) will “see” a higher correlation than the equity investor. For  $d = 1$  this simply means that  $g$  must be decreasing.

While the approach (29) – even in its one-factor version – is certainly capable of generating strong correlation skews, it has a number of drawbacks. First, casual inspections of empirical equity correlations suggests that there is little empirical support for the basic conjecture, namely that firms with high default risk are less correlated to the overall market than firms with low default risk; second, to reproduce the correlation skews observed in practice, the function  $g$  turns out to be quite extreme, producing correlation matrices with elements that get unrealistically close to 0 or 1; and third, the model predicts that near-homogeneous baskets should have no correlation skew (which is unlikely). Furthermore, depending on the details of the chosen specification, one will have the problem that the correlations change daily when spreads are updated. Finally, the extreme correlations required to match observed skews lead to hedges in terms of single-obligor positions which are radically different from those computed with other models.

## 6 NUMERICAL RESULTS

Having previously examined large-portfolio limits, we now turn to a numerical examination of tranches in finite-sized portfolios, using the models and algorithms developed in Sections 3 and 4. While we shall conclude this section with some data for a realistic portfolio (I-Boxx) under real market conditions, to facilitate replication of our results we use a simplified portfolio for most of our tests. Results for more realistic portfolios are qualitatively similar.

We consider a test portfolio consisting of  $N = 25$  obligors each with a notional of 10 million for a total portfolio notional of 250 million. On this portfolio we consider five-year tranches 0–3%, 3–7%, 7–12%, 12–20%, and 20–30%. All models will be calibrated to a market where default-free interest rates are zero, where the default swap spread of the  $i$ th obligor is  $i \cdot 10$  bps, ie, spreads run from 10 to 250 bps, and where the expectation of recovery conditioned on default

<sup>14</sup> Or perhaps a perturbative specification:  $X_i = a_i g(s_i)Z + \varepsilon_i \sqrt{1 - \|a_i g(s_i)\|^2}$ ,  $i = 1, \dots, N$ , where  $a_i$  may be an empirical factor loading, for instance.

$(E(R_i | \tau_i = T))$  is assumed to be 40% for all obligors and all  $T$ . To ease comparison all parameterized models will be required to price the 0–7% base tranche at 20% flat correlation.

To ease the notation we shall refer to the Gaussian copula model with flat correlation  $\rho$  as ‘ $G_\rho$ ’, to the specific random recovery model of Section 3.2 as ‘RR’ and to the specific random factor loading model of Section 4.1 as ‘RFL’.

### 6.1 Random recovery model

For the specific RR model of Section 3.2 we consider a generic model with two factors in order to allow control over the codependence between defaults and recovery rates. We assume that the parameters,  $a$ ,  $b$ , and  $\sigma_\xi$  are obligor-independent and we suppress the obligor index  $i$ . For given values of  $a$ ,  $b$  and  $\sigma_\xi$  we calibrate  $\mu$  such that  $E(R_i | \tau_i = T) = 40\%$ , for all  $T$ , as required. Note that this requires  $\mu$  to depend on  $T$ . Next, for given  $b$  and  $\sigma_\xi$  we choose the value of  $a$  such that the requirement on the valuation of the 0–7% tranche is satisfied.<sup>15</sup>

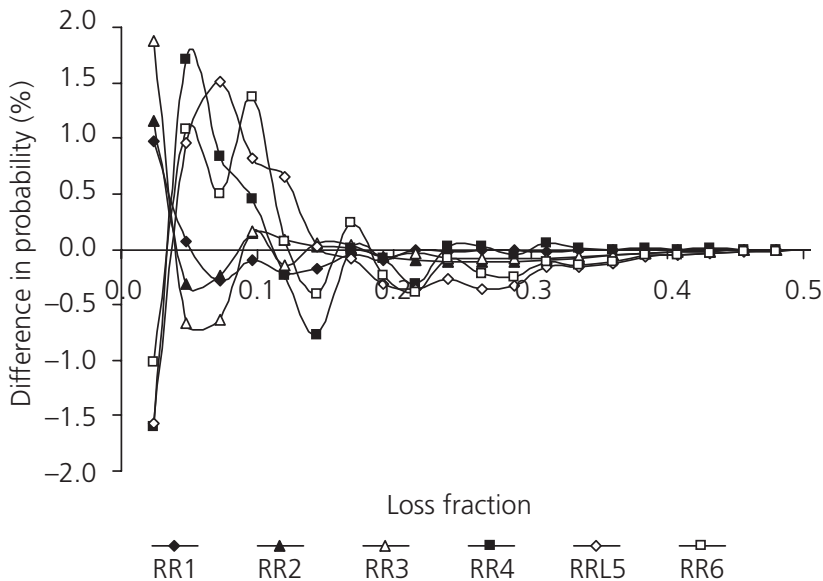
We shall look at the six different sets of parameter values listed in Table 1. From the table we see that the value of  $a$  is smaller in all cases than the  $\sqrt{20\%} \approx 0.447$  required to satisfy the 0–7% tranche price requirement in the absence of random recovery. Going into more detail, we note that the required value of  $a$  increases with  $\sigma_\xi$  and decreases with default-recovery correlation (as measured by  $a \cdot b$ ). Intuitively, this is because tranches are actually sensitive to the covariance between single-obligor *losses* (not just defaults); a large value of  $b$  results in large recovery covariance and this leads to high loss correlation even when default correlations and default-recovery correlations are low. When tranche values are translated into a Gaussian copula framework the loss covariance has to be represented solely in terms of default correlation, which will therefore appear higher. These observations hold also for more realistic tranching portfolios such as I-Boxx.

As pointed out in the introduction, a skew in tranche values arises from a loss

**TABLE 1** The six named sets of parameter values for the RR model. Note that  $a$  and  $b$  are two-dimensional vectors reflecting the fact that the model has two factors.

Name	$a$	$b$	$\sigma_\xi$
RR1	(0.225,0)	(0.8,0)	1.0
RR2	(0.305,0)	(0.4,0.4)	1.0
RR3	(0.355,0)	(0,0.8)	1.0
RR4	(0.195,0)	(0.8,0)	0.01
RR5	(0.3,0)	(0.4,0.4)	0.01
RR6	(0.355,0)	(0,0.8)	0.01

<sup>15</sup> By rotational invariance we can take the second component of  $a$  to be zero.

**FIGURE 6** Loss distributions for random recovery model.

Parameter sets RR1–RR6 are in Table 1. The graphs show the *differences* between the model distributions and the distribution of the  $G_{20\%}$  model.

distribution which differs from the loss distribution of a Gaussian copula model by having lower probability of zero loss and a fat upper tail. Based on our intuitive understanding of the impact of random recovery in generating loss covariance, we would expect that its effect on the loss distribution can largely be compensated by a reduction in default correlation. Thus we would expect a properly based comparison between models with and without random recovery to show little difference.

To examine this in detail, Figure 6 shows the *difference* between the cumulative loss distributions for the RR model with the parameter values of Table 1 and the distribution of the  $G_{20\%}$  model. As expected, the differences are quite small and do not exhibit the characteristics required to generate a skew in tranche values.

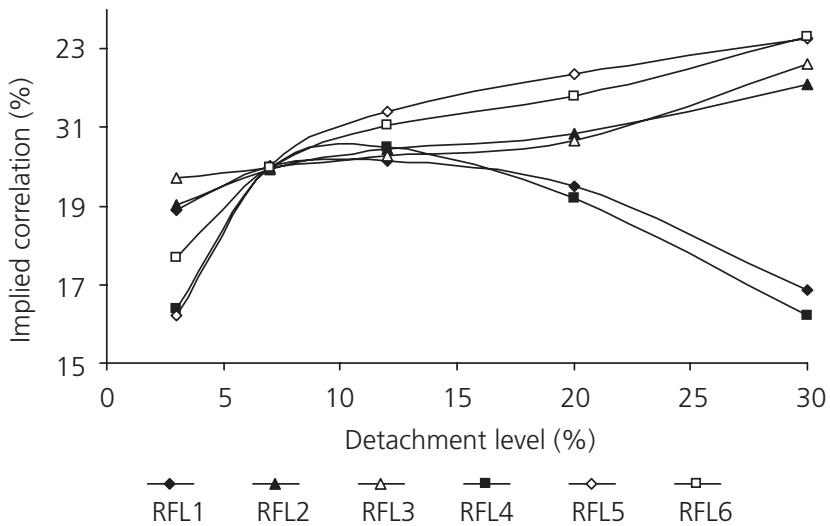
To investigate the impact on tranche values in more detail, Figure 7 shows the implied base correlations for the RR model. We observe that, as expected from the loss distributions, the RR model is not capable of producing strong skews.<sup>16</sup> This finding also applies to tranches of traded index portfolios.

## 6.2 Random factor loadings model

For the specific RFL model of Section 4.1 we shall consider a generic one-factor model where the parameters  $\alpha$ ,  $\beta$  and  $\theta$  are identical for all obligors. We shall look at the six different sets of parameter values listed in Table 2. For given values of

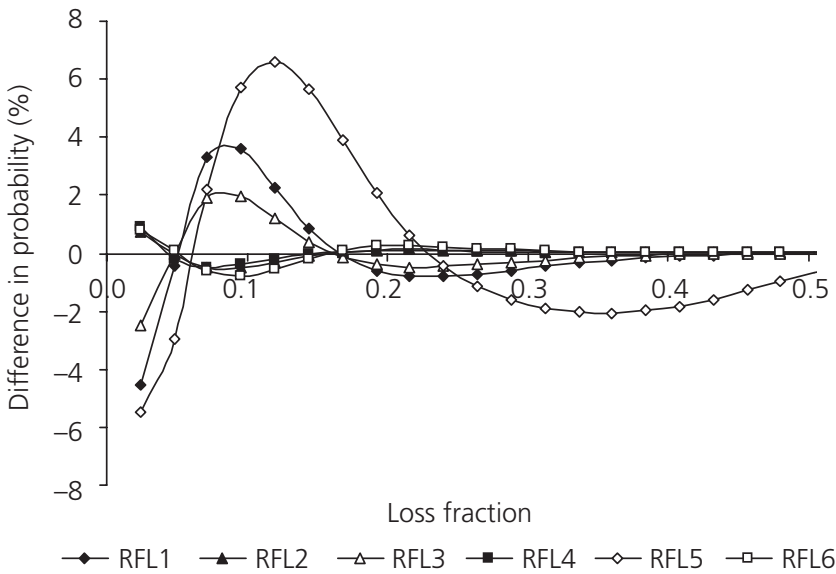
<sup>16</sup>For the special case of independence between default and recovery, this conclusion can also be drawn from numerical results in Gregory and Laurent (2004).

**FIGURE 7** Implied base correlations for random recovery model.



Parameter sets RR1–RR6 are in Table 1.

**FIGURE 8** Loss distributions for random factor loading model.



Parameter sets RFL1–RFL6 are in Table 2. The graphs show the *differences* between the model distributions and the distribution of the  $G_{20\%}$  model.

**TABLE 2** The six named sets of parameter values for the RR model.

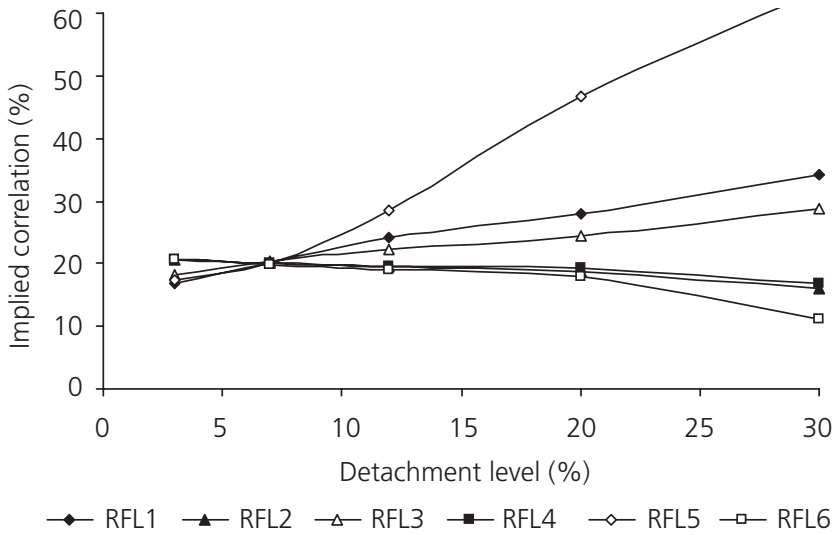
Name	$\alpha$	$\beta$	$\theta$
RFL1	0.62	0.03	0
RFL2	0.425	0.5	0
RFL3	0.54	0.03	1
RFL4	0.43	0.54	1
RFL5	0.9	0.269	-2
RFL6	0.425	0.485	-1

$\beta$  and  $\theta$  we have determined  $\alpha$  such that the requirement on the 0–7% tranche is satisfied. Note that when  $\beta$  is small and  $\theta$  is low (negative) a high value of  $\alpha$  is required. Thus, intuitively, this gives a model where factor loadings are “normally” low, but, for very low factor values, they become very high. In this model senior tranches derive a great deal of value from the many expected defaults in the “disaster state” with high factor loadings, whilst junior tranches derive value from the low effective correlation in the normal state; intuitively, this should produce a strong skew. On the other hand, when  $\theta$  is high (and positive), the difference  $|\alpha - \beta|$  cannot be too great and this leads to more “peaceful” and less skewed models. We first look at the cumulative loss distributions generated by the model with these parameters. To show the differences from the Gaussian copula model more clearly, Figure 8 actually shows the *difference* between the model distributions and the distribution of the  $G_{20\%}$  model. As expected on intuitive grounds, the RFL model can give rise to loss distributions which differ markedly from the distribution of the Gaussian copula model in the way required to generate a skew: lower probability of zero loss and a fat upper tail.

Next, Figure 9 shows the implied base correlations for the same sets of parameters. As is clear from Figure 9, the RFL model is capable of producing strong skews. In particular, we note that the skew is upward (downward) sloped for  $\alpha$  greater (smaller) than  $\beta$  and that the steepest skew is produced with the smallest value (-2) of  $\theta$ . This is not unexpected, given the interpretation of models with small  $\beta$  and low  $\theta$  given above.

We now turn briefly to the properties of the skew, in particular its dynamics under spread changes. The question of skew dynamics is important because it affects the sensitivities of tranche values to default swap spreads. More precisely, when spreads go up, tranche values move for two reasons: first, because of the increase in overall expected loss; second, because (implied) correlations change when the skew changes. It is important to realize that the RFL model does not uniquely determine the skew dynamics; for this, additional specifications are needed. One way of seeing this is to note that there are actually *two* ways of effecting an increase in spreads: either by increasing the default thresholds of indi-

**FIGURE 9** Implied base correlations for random factor loading model.



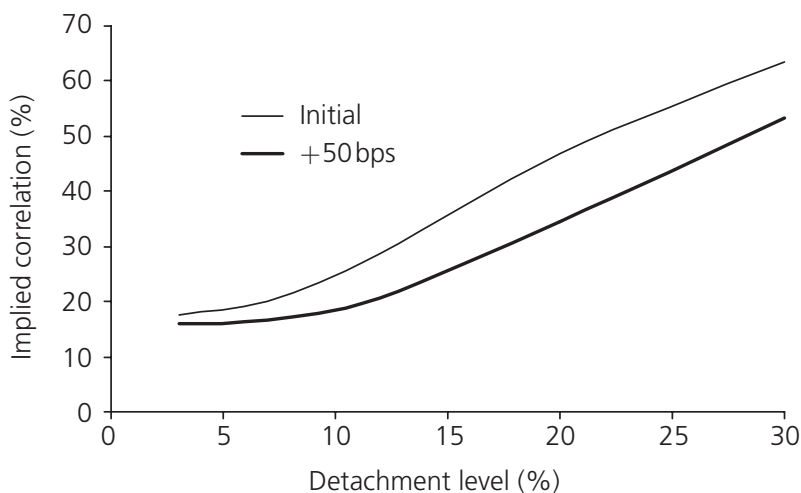
Parameter sets RFL1–RFL6 are in Table 2.

vidual obligors (which is equivalent to decreasing the expectation value of  $\varepsilon_i$ ) or by decreasing the expectation value of the factor  $Z$ . In the Gaussian copula model these two would be equivalent, but in the RFL model this is clearly not so, due to the presence of the non-constant loading functions. The additional specifications required to determine the skew dynamics is equivalent to a specification of how to decompose a given change in default spreads into a shift of default thresholds and a shift in factor mean.

It is easy to see that – when factor loadings are decreasing in the factor value – a spread *increase* effected exclusively by a decrease in the mean of  $Z$  leads to an *upward* shift in the skew. This is because we can equivalently think of a downward shift in the factor mean as a right translation of the factor loadings as functions of the factor. Such a translation will increase the factor loadings for all values of the factor and thus increase implied correlations.

To illustrate what happens when a spread increase is effected exclusively by changing the default thresholds we give in Figure 10 the skews produced by the model with parameters RFL5 when all default swap spreads are increased by 50 bps and the factor mean is kept fixed. When compared to the skew produced for the initial spreads using the same parameters, we see that the effect of such an increase in spreads is to shift the skew to the right. For an upward sloping skew this implies that the base correlation for given detachment level decreases – the opposite of what happens when we shift the factor mean.

As just demonstrated, the RFL model can generate non-trivial skew dynamics. The specifics of the skew dynamics – and thereby of credit spread hedges

**FIGURE 10** Skew dynamics in random factor loading model.

Base correlation for the RFL model with parameters RFL5 after a 50 bps additive shift to all default swap spreads (fat curve). The correlations produced with the initial spreads is given for reference (thin curve).

**TABLE 3** The six named sets of RFL model parameters applied to the tranches of I-Boxx NA.

Name	$\alpha$	$\beta$	$\theta$
RFL7	0.58	0.38	-1.6
RFL8	0.85	0.45	-3
RFL9	1.01	0.41	-2.5
RFL10	1.26	0.36	-2.4
RFL11	0.63	0.01	-0.6

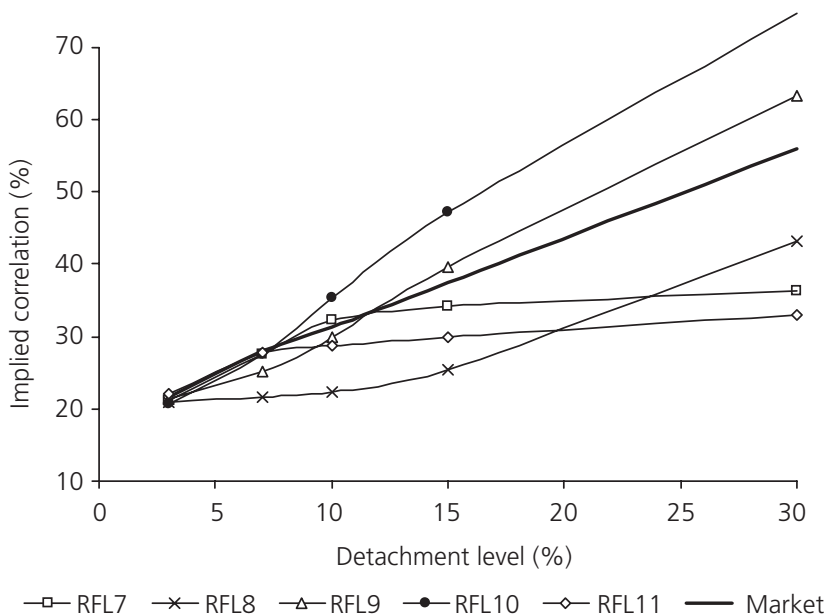
– depend on additional parameters specifying how spread changes affect model parameters<sup>17</sup> (default thresholds and factor mean).

### 6.3 A real-life example: I-Boxx

For a more realistic application of the RFL model we here present the base correlation skews produced for the I-Boxx NA portfolio (of May 2004) with a range of model parameters. The parameter sets are listed in Table 3 and the resulting base correlations are shown in Figure 11. Evidently, the RFL model – even in its

<sup>17</sup> Note that this is not specific to the RFL model – a similar issue arises in, say, the Marshall-Olkin copula – and reflects a well-known issue in hedging using a model with a skew: the sensitivities often depend on assumptions about skew dynamics which cannot be derived from the model itself. See Andersen and Sidenius (2004) for more details.

**FIGURE 11** Base correlation skews for random factor loading model.



Parameter sets RFL7–RFL11 are in Table 3. The market correlation skew is given for reference (fat curve).

simplest form with only three parameters – is capable of generating skews of a wide range of slopes and curvatures for I-Boxx. We note that the market skew is compatible with a low, negative threshold and a large difference in factor loadings between the low-factor and high-factor regimes. It is thus possible that the relatively high premiums of senior tranches are due to a market perception of the existence of a “disaster state” in which not only default probabilities but also effective correlations are high. On the flip side, the relatively high price of insurance on the equity tranche could be due to the perception that correlations are “normally” rather low.<sup>18</sup>

We round off our discussion by again emphasizing that the simple parameterization used in Figure 11 is primarily for illustrative purposes and may not be smooth or rich enough for practical trading. However, we find that it is not very difficult to come up with accurate parameterizations: for instance, if we simply extend the model above to have two (rather than one) threshold levels  $\theta$ , we find that the model can fit liquid indices (such as I-Boxx NA and I-Tracx EU) to within 1–2 bps on the break-even spreads of all quoted tranches. The topic of the “best” RFL model parameterization is a topic for future research.

<sup>18</sup> This perception does not necessarily conflict with the correlation values computed from historical data (such as, eg, the KMV data sets); if the sampling period spans several business cycles, the resulting correlations will be weighted averages of correlations in bear and bull markets.

## 7 CONCLUSION

This paper has introduced two tractable extensions of the Gaussian copula, both aimed at making the model conform better with observed phenomena. The first of these mechanisms – *randomized recovery* – is shown to produce a heavy upper tail in portfolio loss distributions and as such should be of interest in the tail risk management of default-sensitive portfolios. Convenient large-portfolio results useful for this exercise were produced. For finite portfolios, efficient numerical techniques were developed to handle random recovery; we use these techniques, along with a carefully developed calibration mechanism, to demonstrate that it is unlikely that random recovery is the prime mechanism behind the correlation skews observed in CDO tranches on standardized index portfolios. Our second extension – *randomized factor loadings* – does a much better job in this respect and, at reasonable parameter levels, is capable of producing correlation skews similar to those observed in the market. The resulting model is straightforward to parameterize and numerically efficient.

As hinted at earlier, the specific parameterizations and model examples used in this paper were rather simplistic, and work remains in uncovering parameters and functional forms that best describe the market. We note that while we, for clarity, have treated the two model extensions in separation, combining both into a single model is an obvious possibility that may further increase realism (at the expense of more parameters). Other topics for future research involves a deeper examination of the hedges produced by various models, and possible further model extensions to better capture observed term structures of loss distributions. For instance, it is not inconceivable that the RFL loading functions should be a function of the horizon  $T$ , perhaps  $a(Z) = a(Z\sqrt{T})$  as a diffusion-type model might imply. We should note that the Gaussian copula is known to have somewhat problematic temporal properties (see Rogge and Schönbucher, 2003); these would likely be inherited by both extensions covered in this paper.

We conclude the paper with a word of warning about implied correlation skew modeling. Here, it should be kept in mind that the market is in its absolute infancy and all modeling attempts are necessarily based on a very limited set of observations that may not even be fully representative. For instance, it is not inconceivable that market imperfections and imbalances currently contribute effects that are transitory and will abate as the market matures. Parameters and models will then obviously require revisions over time. In the same vein, as time progresses more information will be revealed about the dynamics of the correlation skew and its dependence on spread levels and deal maturity. This, in turn, will allow for evaluation of the realism and hedge performance of models and will undoubtedly lead to further extensions and more sophisticated models. At the current time, however, one is probably best served with relatively simple mechanisms, such as those presented in this paper.

**APPENDIX – SOME GAUSSIAN INTEGRALS**

In our notation  $\varphi$ ,  $\Phi$  and  $\Phi_2$  denote respectively the standard Gaussian density function, the standard Gaussian cumulative distribution function and the standard bivariate Gaussian cumulative distribution function.

LEMMA 1 For arbitrary real constants  $a$ ,  $b$  and  $c$

$$\int_{-\infty}^{\infty} \Phi(ax + b)\varphi(x)dx = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right) \tag{30a}$$

$$\int_{-\infty}^{\infty} \Phi(ax + b)^2 \varphi(x)dx = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}; \frac{a^2}{\sqrt{1+a^2}}\right) \tag{30b}$$

$$\int_{-\infty}^c \Phi(ax + b)\varphi(x)dx = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}\right) \tag{30c}$$

PROOF Consider a Gaussian random variable

$$\tilde{z}_1 \sim N\left(0, \sqrt{1+a^2}\right)$$

We write  $\tilde{z}_1 = -a\tilde{x} + \delta_1$  where  $\tilde{x}, \tilde{\delta}_1 \sim N(0, 1)$  are independent. We notice that for some arbitrary constant  $b$ ,

$$\text{Prob}(\tilde{z}_1 \leq b) = \Phi\left(b/\sqrt{1+a^2}\right)$$

But, by the law of iterated expectations,

$$\begin{aligned} \text{Prob}(\tilde{z}_1 \leq b) &= \text{E}\left(1_{\tilde{z}_1 \leq b}\right) = \text{E}\left(\text{E}\left(1_{\tilde{z}_1 \leq b} \mid \tilde{x}\right)\right) \\ &= \text{E}\left(\text{Prob}(\tilde{\delta}_1 \leq a\tilde{x} + b)\right) = \int_{-\infty}^{\infty} \Phi(ax + b)\varphi(x)dx \end{aligned}$$

This proves (30a). Introduce now a second variable

$$\tilde{z}_2 = N\left(0, \sqrt{1+a^2}\right)$$

and write  $\tilde{z}_2 = -a\tilde{x} + \delta_2$ , where  $\tilde{x}$  is defined above and  $\tilde{\delta}_2 \sim N(0, 1)$  is independent of both  $\tilde{x}$  and  $\tilde{\delta}_1$ . The correlation between  $\tilde{z}_1$  and  $\tilde{z}_2$  is  $a^2/(1+a^2)$ , such that

$$\text{Prob}(\tilde{z}_1 \leq b, \tilde{z}_2 \leq b) = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}; \frac{a^2}{\sqrt{1+a^2}}\right)$$

but also,

$$\begin{aligned} \text{Prob}(\tilde{z}_1 \leq b, \tilde{z}_2 \leq b) &= \text{E}(\text{Prob}(\tilde{z}_1 \leq b, \tilde{z}_2 \leq b | \tilde{x})) \\ &= \text{E}(\text{Prob}(\tilde{z}_1 \leq b | \tilde{x}) \text{Prob}(\tilde{z}_2 \leq b | \tilde{x})) \end{aligned}$$

where we have relied on conditional independence in the second equality. Recognizing that  $\text{Prob}(\tilde{z}_1 \leq b | \tilde{x}) = \text{Prob}(\tilde{z}_2 \leq b | \tilde{x}) = \Phi(a\tilde{x} + b)$  proves (30b). Finally, to prove (30c) note that

$$\begin{aligned} \text{Prob}(\tilde{z}_1 \leq b, \tilde{x} \leq c) &= \int_{-\infty}^c \text{Prob}(\tilde{z}_1 \leq b | \tilde{x} = x) \varphi(x) dx \\ &= \int_{-\infty}^c \Phi(ax + b) \varphi(x) dx \end{aligned}$$

On the other hand,

$$\text{Prob}(\tilde{z}_1 \leq b, \tilde{x} \leq c) = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}\right)$$

where we have used the fact that the correlation between  $\tilde{z}_1$  and  $\tilde{x}$  is  $-a/\sqrt{1+a^2}$ . □

An expression for the integral in (30b), but with a finite upper integration limit is also possible, but involves the three-dimensional Gaussian distribution and is likely easier to compute numerically.

LEMMA 2 Let  $x$  be a standard Gaussian variate. For constants  $a$  and  $b$  define

$$\omega := \frac{a}{\sqrt{1+a^2}}$$

Then

$$\text{E}(x\Phi(ax+b)) = \frac{\omega e^{-\frac{1}{2}b^2(1-\omega^2)}}{\sqrt{2\pi}} = \omega \varphi(b\sqrt{1-\omega^2}) \tag{31}$$

PROOF Note that

$$\begin{aligned} \text{E}(x\Phi(ax+b)) &= \int_{-\infty}^{\infty} x\varphi(x)\Phi(ax+b)dx = - \int_{-\infty}^{\infty} \varphi'(x)\Phi(ax+b)dx \\ &= a \int_{-\infty}^{\infty} \varphi(x)\varphi(ax+b)dx - [\varphi(x)\Phi(ax+b)]_{-\infty}^{\infty} \\ &= \frac{a}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+a^2x^2+2abx+b^2)} dx \\ &= \frac{a}{2\pi} e^{-\frac{1}{2}b^2(1-\omega^2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x\sqrt{1+a^2}+b\omega)^2} dx \end{aligned}$$

Clearly, then,

$$E(x\Phi(ax+b)) = \frac{a}{2\pi} e^{-\frac{1}{2}b^2(1-\omega^2)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1+a^2}} e^{-\frac{1}{2}y^2} dy \quad \square$$

LEMMA 3 Let  $x$  be a standard Gaussian variate. For constants  $a$  and  $b$

$$E(x^2\Phi(ax+b)) = \frac{-b\omega}{\sqrt{1+a^2}} E(x\Phi(ax+b)) + \Phi\left(\frac{b}{\sqrt{1+a^2}}\right) \quad (32)$$

PROOF We have first

$$\begin{aligned} E(x^2\Phi(ax+b)) &= \int_{-\infty}^{\infty} x^2\varphi(x)\Phi(ax+b)dx \\ &= \int_{-\infty}^{\infty} (\varphi''(x) + \varphi(x))\Phi(ax+b)dx \end{aligned}$$

Here

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi''(x)\Phi(ax+b)dx &= -a \int_{-\infty}^{\infty} \varphi'(x)\varphi(ax+b)dx + [\varphi'(x)\Phi(ax+b)]_{-\infty}^{\infty} \\ &= a \int_{-\infty}^{\infty} x\varphi(x)\varphi(ax+b)dx \end{aligned}$$

and

$$\begin{aligned} a \int_{-\infty}^{\infty} x\varphi(x)\varphi(ax+b)dx &= \frac{a}{2\pi} e^{-\frac{1}{2}b^2(1-\omega^2)} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}(x\sqrt{1+a^2}+b\omega)^2} dx \\ &= \frac{a}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2(1-\omega^2)} \int_{-\infty}^{\infty} \left(\frac{y-b\omega}{\sqrt{1+a^2}}\right) \frac{1}{\sqrt{2\pi}\sqrt{1+a^2}} e^{-\frac{1}{2}y^2} dy \\ &= \frac{-b\omega^2\varphi\left(\frac{b\sqrt{1-\omega^2}}{\sqrt{1+a^2}}\right)}{\sqrt{1+a^2}} \end{aligned}$$

The result now follows from Lemmas 1 and 2. □

LEMMA 4 Let  $x$  be a standard Gaussian variate. For constants  $a$  and  $b$  define

$$\omega := \frac{a}{\sqrt{1+a^2}} \quad \text{and} \quad v := \frac{\omega}{\sqrt{1+\omega^2}}$$

Then

$$\begin{aligned} E(x^2\Phi(ax+b)^2) &= \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}; \frac{a^2}{\sqrt{1+a^2}}\right) \\ &+ \frac{2E(x\Phi(ax+b))}{\sqrt{1+a^2}} \left[ v\varphi\left(b(1-\omega^2)\sqrt{1-v^2}\right) - b\omega\Phi\left(\frac{b(1-\omega^2)}{\sqrt{1+\omega^2}}\right) \right] \end{aligned}$$

PROOF First

$$\begin{aligned} E\left(x^2\Phi(ax+b)^2\right) &= \int_{-\infty}^{\infty} x^2\varphi(x)\Phi(ax+b)^2 dx \\ &= \int_{-\infty}^{\infty} (\varphi''(x) + \varphi(x))\Phi(ax+b)^2 dx \end{aligned}$$

Here

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi''(x)\Phi(ax+b)^2 dx \\ &= -2a \int_{-\infty}^{\infty} \varphi'(x)\Phi(ax+b)\varphi(ax+b)dx + \left[\varphi'(x)\Phi(ax+b)^2\right]_{-\infty}^{\infty} \\ &= 2a \int_{-\infty}^{\infty} x\varphi(x)\Phi(ax+b)\varphi(ax+b)dx \\ &= \frac{2a}{2\pi} e^{-\frac{1}{2}b^2(1-\omega^2)} \int_{-\infty}^{\infty} x\Phi(ax+b)e^{-\frac{1}{2}(x\sqrt{1+a^2}+b\omega)^2} dx \\ &= \frac{2E(x\Phi(ax+b))}{\sqrt{1+a^2}} \int_{-\infty}^{\infty} (y-b\omega)\Phi(y\omega+b(1-\omega^2))\varphi(y)dy \end{aligned}$$

We can now use Lemma 2 to see that

$$\int_{-\infty}^{\infty} y\Phi(y\omega+b(1-\omega^2))\varphi(y)dy = v\varphi\left(b(1-\omega^2)\sqrt{1-v^2}\right)$$

and the lemma follows by application of Lemma 1. □

LEMMA 5 For a standardized Gaussian variable  $x$  and arbitrary constants  $a$  and  $b$ , we have

$$E\left(1_{a < x \leq b} x\right) = 1_{b \geq a} (\varphi(a) - \varphi(b)) \tag{33a}$$

$$E\left(1_{a < x \leq b} x^2\right) = 1_{b \geq a} (\Phi(b) - \Phi(a)) + 1_{b \geq a} (a\varphi(a) - b\varphi(b)) \tag{33b}$$

In particular,

$$E(1_{x \leq b} x) = -\varphi(b); E(1_{x > a} x) = \varphi(a); E(1_{x > a} x^2) = a\varphi(a) + (1 - \Phi(a))$$

and

$$E(1_{x \leq b} x^2) = \Phi(b) - b\varphi(b).$$

PROOF If  $b < a$ , the expectations (33a)–(33b) are obviously zero, consistent with the given expressions. For  $b \geq a$ , we have, trivially,

$$E\left(1_{a < x \leq b} x\right) = 1_{b > a} \int_a^b x\varphi(x)dx = -1_{b > a} \int_a^b \varphi'(x)dx = 1_{b > a} (\varphi(a) - \varphi(b))$$

and

$$\begin{aligned} E(1_{a < x \leq b} x^2) &= 1_{b > a} \int_a^b x^2 \varphi(x) dx = 1_{b > a} \int_a^b (\varphi''(x) + \varphi(x)) dx \\ &= 1_{b > a} (\Phi(b) - \Phi(a)) + 1_{b > a} (a\varphi(a) - b\varphi(b)) \quad \square \end{aligned}$$

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