Surprising pricing

Samuel Chiu and Jean-Pascal Crametz show that when two no-arbitrage relationships in bandwidth pricing are combined, opportunities for arbitrage may still exist

With the dramatic rise in both the demand and supply of bandwidth, determining the appropriate price of capacity and managing the accompanying risks have become increasingly important for the telecoms industry. This article examines two no-arbitrage relationships governing the price of bandwidth: temporal arbitrage and geographical arbitrage. Both are unique to bandwidth trading.

We shall first derive the dynamic pricing relationship between time derivatives (forward prices) and the static geographical pricing relationship between related routes to illustrate the pricing constraints. We show that, even though the pricing relationship individually satisfies the two no-arbitrage conditions, arbitrage opportunities may still exist when the two arbitrage features are taken jointly (which we will refer to as composite arbitrage). This observation suggests that the design of bandwidth derivatives must specifically incorporate the special features of telecoms capacity, distinct from that of other well-traded commodities.

Arbitrage can loosely be defined as the opportunity to trade to generate a risk-free profit. Many derivative pricing formulas are derived by relating the prices of the derivatives to their underlying assets, based on the no-arbitrage argument (using various forms of the Black-Scholes options pricing formula, for example). We have already identified several no-arbitrage conditions specific to bandwidth trading as a commodity.[1]

Temporal arbitrage

Bandwidth contracts are typically characterised, among other legal and financial considerations, by the following parameters:

- Route (city pair)
- Throughput
- Contract duration
- Monthly (constant) payment
- Quality-of-service requirements

We will examine only contract duration and monthly payment, keeping other parameters fixed. We will also consider spot bandwidth contracts. The following schematic payment streams show contracts of various durations. Note the constant monthly payment common in bandwidth contracts:

A. Spot and forward prices

t =	0	1	2	3	4	5	6
Spot, p:	100	99.5	99.75	99.75	99.5	99.25	99.1
Forward, f:	100	98.99583	100.2563	99.75	98.47899	97.9684	98.17338

p_0
p_1, p_1
p_2, p_2, p_2
p_3, p_3, p_3, p_3
$p_n, p_n, p_n, p_n, \ldots\ldots\ p_n, p_n$

For example, p_0 is the spot price for a (current) one-month contract and p_3, p_3, p_3, p_3 is the constant monthly payment stream for a four-month spot contract.

A simple derivative implied by these spot contracts of various durations is the forward contract: How much should I pay for one-month capacity to be delivered n months from now? We will use the notation f_n to represent the price of one-month capacity delivered n months from now. The following payment schemes should be equivalent, such that there will be no arbitrage opportunity:

$p_n, p_n, p_n, p_n, \ldots\ldots\ p_n, p_n$ and
$f_0, f_1, f_2, f_3, \ldots\ldots\ f_{n-1}, f_n$

It should be clear that $f_0 = p_0$. We will make the usual idealised conditions of a "frictionless" market (no transaction costs, no taxes, etc) when we use the no-arbitrage argument to price the f_ts from the p_ts. We first argue for a formula to compute f_1, and then provide a general formula to compute subsequent f_ts.

We first note that one can replicate a two-month contract (month 0 and month 1) by buying a one-month spot and a one month forward. The value (or total discounted payment) of these two alternatives should equal each other:

(p_1, p_1) should be equivalent to $(f_0, f_1) = (p_0, f_1)$, since $f_0 = p_0$

The value (discounted present value) of (p_1, p_1) can be computed as:

$$p_1 + d_{10}\, p_1 = (1 + d_{10})\, p_1,$$

where d_{10} is the discount factor from month 1 to the present. We will not discuss the discount factors in this paper. The value of $(f_0, f_1) = (p_0, f_1)$ can be similarly evaluated as:

$$p_0 + d_{10}\, f_1 .$$

Equating these two values gives:

$$f_1 = [(1 + d_{10})\, p_1 - p_0]/d_{10}.$$

To derive forward prices f_ts from the p_ts, we introduce the following notations:

d_{t0} = discount factor from month t to the present

ds_{t0} = sum of discount factors = $d_{00} + d_{10} + d_{20} + \ldots. + d_{t0}$, note that $d_{00} = 1$

$\mathbf{P_t}$ = a $(t+1)$ dimensional vector of the p_ts = $(p_t, p_t, p_t, p_t, \ldots\ldots\ p_t, p_t)$

$\mathbf{F_t}$ = a $(t+1)$ dimensional vector of the f_is = $(f_0, f_1, f_2, f_3, \ldots\ldots\ f_{t-1}, f_t)$

$\mathbf{D_t}$ = a $(t+1)$ dimensional vector of the d_{i0}s = $(d_{00}, d_{10}, d_{20}, d_{30}, \ldots\ldots\ d_{t0})$

$\mathbf{1}$ = vector of all 1s = $(1, 1, 1, \ldots., 1)$, with dimension dictated by context

V_t = value of the contract $(p_t, p_t, p_t, p_t, \ldots\ldots\ p_t, p_t) = \mathbf{D_t}{*}\mathbf{P_t} = (ds_{t0})(p_t)$

Note that * denotes the dot product.

One can replicate the contract $\mathbf{P_t}$ by buying a contract $\mathbf{P_{t-1}}$, augmented by a one-month forward f_t. Therefore:

$$V_t = V_{t-1} + d_{t0}\, f_t,$$

implying,

$$f_t = [V_t - V_{t-1}]/d_{t0} = [\mathbf{D_t}{*}\mathbf{P_t} - \mathbf{D_{t-1}}{*}\mathbf{P_{t-1}}]/d_{t0} = [(ds_{t0})(p_t) - (ds_{t-1,\,0})(p_{t-1})]/d_{t0} \qquad (1)$$

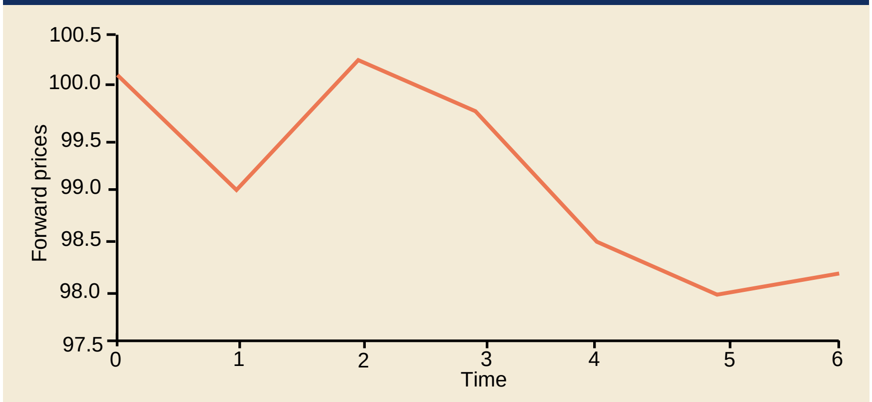
Therefore, f_t can be derived using the two consecutive contract payment terms p_{t-1} and p_t (and the appropriate discount factor). These calculations are identical to those used in the computations of the Rate Xchange's Revealed Price Index (RPI) and the Revealed Forward Price Index (RFP)[2], when all discount factors are taken to be one (zero-interest). This replication (or comparison) principle can be used to construct a risk-free profit opportunity if the forward prices do not follow the above equations.

It is also possible to construct the contract

[1] *Pricing Bandwidth Derivatives (1999) by Barz, Chiu, Crametz and Mayfield and Financial Instruments in a Telecommodity Exchange Market (1999), available at http://www.ratxlabs.com/research.html*
[2] *These can be found at www.ratexlabs.com*

Arbitrage relationships

1. The forward curve



price, the p_ts, by using the forward price vector $\mathbf{F}_t$. The formula is provided below. It is derived by noting the value equivalent of (p_t, p_t, p_t, p_t, ……. p_t, p_t) and (f_0, f_1, f_2, f_3, ……. f_{t-1}, f_t).

$$p_t = [\mathbf{D}_t{*}\,\mathbf{F}_t]/ ds_{t0} \qquad (2)$$

Equations (1) and (2) are particularly suitable for spreadsheet implementation, in which it is possible to experiment with various sensitivity and optimisation schemes. Table A assumes an annual interest rate of 10% (or a monthly rate of 10/12 %). The forward price is derived using equation (1). The spot price is a function of contract duration t (see table A). This calculation shows a perceived bandwidth shortage in months 2 and 3. The forward curve is plotted in figure 1.

Geographical arbitrage

Geographical arbitrage consideration for bandwidth pricing is a consequence of the network topology. It is a simple condition that dictates that bandwidth between a city pair cannot be priced strictly higher than any two (or more) routes connecting the same city pair. For example, the price of bandwidth from New York to Los Angeles cannot cost more than the bandwidth connecting New York and Houston plus that connecting Houston and Los Angeles. Otherwise, a trader could buy the two connecting pieces and sell it for a risk-free profit. This arbitrage situation can be observed between New York and some European cities, via London, simply because of the lack of price transparency at the present time[3]. The no-arbitrage restriction can be simply expressed as a price "triangular inequality" condition:

$$p^a + p^b >= p^c,\quad p^a + p^c >= p^b \text{ and } p^b + p^c >= p^a \qquad (3)$$

where a, b and c are three sides that form a triangle. These conditions can easily be observed and verified when price information is transparent (as one function of an exchange).

Composite arbitrage

The temporal no-arbitrage conditions in (1) and (2) are straightforward. The geographical no-arbitrage condition in (3) is also transparent once price information is known and other multiple link connections have been checked. We now ask the following question, which is illustrated in figure 2:

If the forward price is dynamically linked to the spot contract prices (of different durations) via (1), thus satisfying the no-temporal arbitrage condition, on each of three routes (a, b and c) forming a triangle, and the spot contract prices satisfy the geographical no-arbitrage condition in (3) for all contract durations, will the forward prices so computed be arbitrage free?

Mathematically, we have a spot price vector for three geographical routes:

p^k_t = spot price for the three (triangle-forming) routes k = a, b, c.

$p^a_t + p^b_t >= p^c_t$, $p^a_t + p^c_t >= p^b_t$ and $p^b_t + p^c_t >= p^a_t$

A set of forward prices (one for each route) is computed using the no-temporal arbitrage equation (1), indexed by route k = a, b, c:

$$f^k_t = [(ds_{t0})(p^k_t) - (ds_{t-1,\,0})(p^k_{t-1})]/d_{t0}$$

Will the set of forward prices $\{f^k_t\}$ satisfy the no-geographical arbitrage inequality?

$f^a_t + f^b_t >= f^c_t$, $f^a_t + f^c_t >= f^b_t$, and $f^b_t + f^c_t >= f^a_t$, for all t?

One would hope for an affirmative answer, since the spot prices satisfy the geographical no-arbitrage condition and the forward price on each route is derived from the spot prices with no temporal arbitrage opportunities. However, the answer is, surprisingly, negative, as the following example shows. We again use an annual interest rate of 10% in this example (see table B). The * entries are not relevant in our computation.

Note that all the p^k_ts satisfy the triangular inequality condition, and that the forward prices are derived (route by route) using the no-temporal arbitrage condition in equation (1). We observe from this example that the forward price 12 months out ($t = 12$) violates the geographical no-arbitrage condition:

$$f^a_{12} + f^b_{12} < f^c_{12} \text{ as}$$
$$82.66 + 73.157 < 164.68 \qquad (4)$$

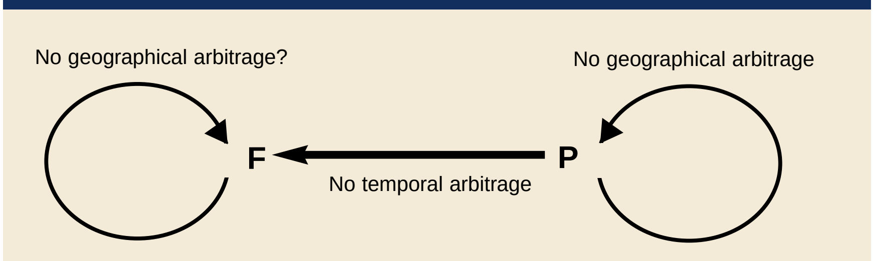
One can presumably buy the two connecting routes b and c and sell the combined route forwards to satisfy route c and realise a risk-free profit of 8.86. The analysis leading to the above example is based on the above inequality (4). Hence the following question: What pricing conditions will present us with an arbitrage opportunity as expressed in (4)? The following logical step provides us with a simple inequality that will imply (4):

$$f^a_t + f^b_t < f^c_t,$$

Substituting with equation (1):

$$[(ds_{t0})(p^a_t) - (ds_{t-1,\,0})(p^a_{t-1})]/d_{t0} + [(ds_{t0})(p^b_t) - (ds_{t-1,\,0})(p^b_{t-1})]/d_{t0}$$
$$< [(ds_{t0})(p^c_t) - (ds_{t-1,\,0})(p^c_{t-1})]/d_{t0}$$

2. Are forward prices arbitrage free?



[3] Data Mining using RateXchange's Lead-Generation Data. Available from the authors.

$[(ds_{t0})(p^a_t) - (ds_{t-1,0})(p^a_{t-1})] + [(ds_{t0})(p^b_t) - (ds_{t-1,0})(p^b_{t-1})]$
$< [(ds_{t0})(p^c_t) - (ds_{t-1,0})(p^c_{t-1})]$

Re-arranging terms:

ds_{t0} [$Slack_t$] < $ds_{t-1,0}$ [$Slack_{t-1}$],
where $Slack_t = p^a_t + p^b_t - p^c_t$ and $Slack_{t-1} = p^a_{t-1} + p^b_{t-1} - p^c_{t-1}$

$Slack_k$ is the slack in the triangular inequality. As guaranteed by the geographical no-arbitrage condition on the price vector ($\mathbf{P^a_t + P^b_t >= P^c_t}$), we know that these slacks are non-negative: $Slack_t >= 0$ and $Slack_{t-1} >= 0$. The non-negativity of the slacks allows us to further re-arrange the terms and arrive at the following simple ratio test, which answers our question:

$$\mathbf{Slack_t / Slack_{t-1} < ds_{t-1,0} / ds_{t0}} \quad (5)$$

Application: In our example, we use a constant, annual interest rate of 10%, implying a monthly rate of r' = 10/12 %. We compute ds_{t0} as:

$ds_{t0} = d_{00} + d_{10} + d_{20} + ... + d_{t0} = 1 + [1/(1+r')] + [1/(1+r')]^2 + ... + [1/(1+r')]^t$
$ds_{11,0} = 11.4693$, and $ds_{12,0} = 12.3745$, with
$ds_{11,0} / ds_{12,0} = 0.92685$.

The two triangular inequality slacks are computed as:

B. An example

t:	Spot, p Route a	Route b	Route c	Forward, f Route a	Route b	Route c
10	110	110	110	*	*	*
11	110	110.75	110	110	119.4242	110
12	108	108	114	82.66	73.157	164.68

$Slack_{12} = p^a_{12} + p^b_{12} - p^c_{12} = 108+108-114 = 102$, and
$Slack_{11} = p^a_{11} + p^b_{11} - p^c_{11} = 110+110.75-110 = 110.75$

The slack ratio is computed to be:

$Slack_{12} / Slack_{11} = 0.92099 < 0.92685 = ds_{11,0} / ds_{12,0}$

Therefore, a geographical arbitrage opportunity exists in the forward price space, as indicated by the derived ratio test in (5) and verified by the numerical example.

Conclusion

Even though the pricing relationship between spots and forwards obey the temporal restrictions to avoid arbitrage and the spot prices are governed by the geographical inequality, the derived forward prices are not necessarily arbitrage-free. This observation suggests that one should be careful in establishing no-arbitrage pricing conditions taking into account the special features underlying bandwidth as a commodity. Much research still needs to be carried out to understand these pricing relationships. More insight on such issues will be gained as trading becomes more liquid in the future. ■

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